LINAC-BASED FREE-ELECTRON LASER

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Abstract
A basic treatment of the principle of the linac-driven free-electron laser (FEL) is given. The first part of the paper describes the FEL in low-gain approximation, the second part gives the high-gain FEL theory. The majority of the treatment describes FELs in one-dimensional approximation. Only in the final section are a few remarks made on the effects by diffraction of radiation and by electron beam emittance. The ambition of this paper is to make a clear presentation of basic FEL theory concepts with explicit derivation of the formulae from first principles, and it does not aspire to postulate any progress in FEL theory.

1 Introduction

The basic theory of linac-driven free-electron lasers (FEL) in this paper is based on lectures given at the CERN Accelerator School on 'Synchrotron Radiation and Free Electron Lasers' 2–9 July 2003 in Brunnen, Switzerland. This paper is neither a report on progress in FEL theory nor a complete and in-depth treatment of the subject. It presents the basic concepts of linac-based FELs starting from first principles and deriving formulae step-by-step so that students can follow without doing long derivations and calculations. In the interest of simplicity, the FEL theory is given in one-dimensional approximation, i.e. only longitudinal electron motion is considered and diffraction effects of radiation are neglected. This approximation is particularly justified for FELs operating in the VUV- or X-ray wavelength regime because of the suppression of space-charge effects at ultra-relativistic energies typical for these kinds of radiation sources.

The paper covers the material presented during the two one-hour lectures. It includes a few useful comments for the student. The MKSA (or 'practical') system of units and a right-handed Cartesian coordinate system (with *z* being the longitudinal coordinate) are used throughout the paper [1,2].

2 The free-electron laser in low-gain approximation

2.1 Radiation power of a point-like electron distribution moving at ultra-relativistic speed

An FEL is basically a classical device; with very few exceptions, all features can be derived and described by classical electrodynamics and relativistic kinematics. Thus, as an introduction to the principle of FELs, it is useful to recall some basics of classical electrodynamics [3].

Consider an electric charge *q* moving at ultra-relativistic speed with respect to the laboratory system. Classical electrodynamics says that any accelerated charge emits electromagnetic radiation. The radiation power P_γ emitted by a charge *q* accelerated at $\dot{v}^*$ is given by

$$P_\gamma = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\dot{v}^*\right)^2 = P_\gamma^* \quad , \tag{1}$$

where ε_0 is the electric permittivity of vacuum and c is the speed of light. The asterisk * means that the respective quantity is to be evaluated in a system * moving along with the charge such that its velocity v^* is much smaller than c.

Equation (1) states that the power P_γ observed in any system is the same as the power P_γ^* calculated in the co-moving system in Eq. (1). This makes it easy to calculate the radiation power observed in the laboratory system in terms of quantities measured in the laboratory system: We just have to express the acceleration $\dot{v}^*$ by quantities measured in the laboratory system. This is accomplished by the Lorentz transformation of acceleration given by (see Ref. [3] p. 47 ff)

$$\dot{v}_z^* = \gamma^3 \dot{v}_z, \quad \dot{v}_y^* = \gamma^2 \dot{v}_y, \quad \dot{v}_x^* = \gamma^2 \dot{v}_x \tag{2}$$

with $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v_0/c$. The velocity v_0 of the moving system * with respect to the laboratory system is assumed to be in the z direction, see Fig. 1.

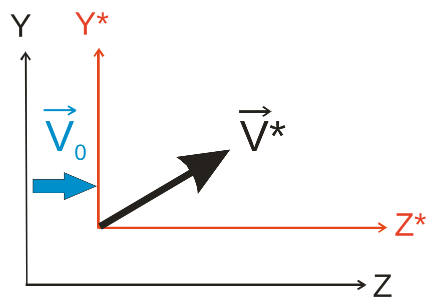

Fig 1: Definition of a coordinate system denoted with an asterisk * moving with speed v_0 with respect to the laboratory system

It is important to realize from Eq. (1) that the component $\dot{v}_z$ of acceleration parallel to the velocity of the moving system transforms in a different way than the components perpendicular to it. Acceleration perpendicular to the relativistic motion of the electron beam is the only one of practical relevance, because it is achieved by the motion of the electrons in the presence of an external magnetic field. In the case of vertical acceleration, for example, Eq. (1) reads

$$P_\gamma = \frac{q^2}{6\pi\varepsilon_0 c^3} \gamma^4 \dot{v}_y^2 , \tag{3}$$

indicating that one gains a large increase in radiation power when accelerating the electron beam to ultra-relativistic (i.e. $\gamma >> 1$) energies.

In terms of the FEL principle, the most important consequence of Eq. (1) is that the radiation power scales quadratically with the charge. Taking into account that, in practice, the charge consists of a large number N of electrons with elementary charge e_0, Eq. (3) can be written in the form

$$P_\gamma = \frac{N^2 e_0^2}{6\pi\varepsilon_0 c^3} \gamma^4 \dot{v}_y^2 . \tag{4}$$

Obviously, the radiation power per electron is $P_\gamma(\text{each electron}) = P_\gamma/N = (N e_0^2/6\pi\varepsilon_0 c^3)\gamma^4 \dot{v}_y^2$, i.e. it is N times larger than the radiation power of a single electron (i.e. not accompanied by many others)

$P_\gamma(\text{single electron}) = (e_0^2/6\pi\varepsilon_0 c^3)\gamma^4 \dot{v}_y^2$. This is because the electrons moving in a bunch have to perform work against the electric field generated by the co-moving electrons. This can be considered the classical analogue to stimulated emission.

The main condition for Eq. (4) to hold is that all N electrons have to make up a 'point-like' charge distribution. For a radiating bunch of electrons moving at ultra-relativistic speed this means that the longitudinal dimension of the bunch must be shorter than the radiated wavelengths. For wavelengths much shorter than the visible, this is difficult (or impossible) to achieve. In conventional synchrotron radiation sources like electron storage rings, for instance, the radiation wavelength attractive for users is in the nanometre range (or below), while the size of electron bunches in storage rings is typically a few millimetres. As a consequence, the radiation power of a bunch of N electrons in a storage ring is only $N \cdot P_\gamma(\text{single electron})$. Electrons radiate independently from one another (incoherent radiation). Obviously, there is a huge N factor to be regained. The FEL principle provides a mechanism to rearrange the electrons on the scale of the optical wavelength.

Figure 2 shows the key components of a free-electron laser using an electron beam accelerated by a linear accelerator.

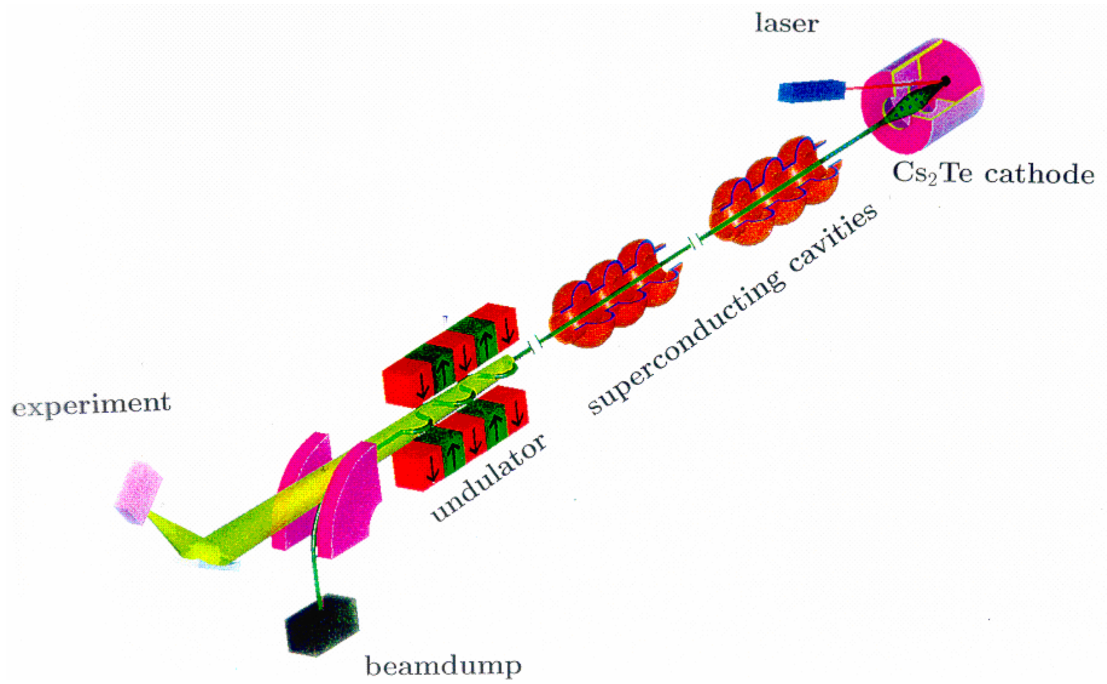

Fig. 2: A linac-driven free-electron laser. Major components are i) a source of electron bunches of high charge density, ii) a linear accelerator (superconducting technology is preferable to achieve a high duty cycle, but is not a must), iii) a long undulator magnet generating periodically alternating deflection of the electron beam, and iv) a bending magnet separating the FEL radiation generated in the undulator from the electron beam.

2.2 Electron motion in the undulator field

In the present paper, we restrict ourselves to helical undulators, because this simplifies calculations. Extension to planar undulators can be found in the literature. It modifies some quantitative results but it does not change essentials.

In the vicinity of the axis of a helical undulator with period length λ_u, the magnetic field can be expressed (to first order in the distance r to the axis) by

$$\vec{\mathbf{B}} = \mathbf{B}\begin{pmatrix} -\sin(k_u z) \\ \cos(k_u z) \\ 0 \end{pmatrix} + O(r^2) \quad \left(\text{using } k_u = \frac{2\pi}{\lambda_u}\right). \tag{5}$$

The equation of motion of the electron in this field is

$$m\gamma\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \vec{\mathbf{B}} = q\mathbf{B}\begin{pmatrix} -\dot{z}\cdot\cos(k_u z) \\ -\dot{z}\cdot\sin(k_u z) \\ \dot{x}\cdot\cos(k_u z) + \dot{y}\cdot\sin(k_u z) \end{pmatrix}. \tag{6}$$

One solution to this equation is a periodic, helical motion:

longitudinal motion: v_z = const., $z = v_z t = \beta_z ct$;

transverse motion on a circle: $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = c\frac{K}{\gamma}\begin{pmatrix} -\sin(k_u z) \\ \cos(k_u z) \end{pmatrix}$,

or, using $w = x + iy$

$$\dot{w} = ic\frac{K}{\gamma}\exp\left(ik_u z\right). \tag{7}$$

This can be solved easily:

$$w = \frac{cK}{\gamma k_u v_z}\exp\left(ik_u z\right) .$$

Here, $K = e\lambda_u B/2\pi m_0 c$ is called the undulator parameter. It is typically $K \approx 1$.

The opening angle of helical motion is seen to be

$$\frac{v_\perp}{c} = \frac{K}{\gamma} \approx \frac{1}{\gamma} .$$

With this result, we can now determine $\beta_z = \frac{v_z}{c}$:

$$\beta_z = \frac{1}{c}\sqrt{v^2 - \dot{x}^2 - \dot{y}^2} = \sqrt{\beta^2 - \left(\frac{K}{\gamma}\right)^2} = \sqrt{1 - \frac{1}{\gamma^2} - \left(\frac{K}{\gamma}\right)^2} \approx 1 - \frac{1}{2\gamma^2}\left(1 + K^2\right) . \tag{8}$$

2.3 Interaction with electromagnetic wave

Consider an external electromagnetic wave moving parallel to the electron beam, i.e. in z direction. Assuming a plane wave, which has zero z component of the electric field vector:

$$\vec{\mathbf{E}}_L = \mathbf{E}_0 \begin{pmatrix} \cos(\omega_L t - k_L z - \varphi_0) \\ \sin(\omega_L t - k_L z - \varphi_0) \\ 0 \end{pmatrix} , \tag{9}$$

with the magnetic field given by: $\vec{\mathbf{B}}_L = (1/c\omega_L)\dot{\vec{\mathbf{E}}}_L$. Here, ω_L is the angular frequency of the electromagnetic 'light' wave and the index L stands for 'light'. It is unnecessary to mention that this frequency does not need to be in the visible range. $k_L = 2\pi/\lambda_L$ is the wave number. Again, complex notation is very useful, because we have to deal with only two components of $\vec{\mathbf{E}}_L$. We define a complex electric field given by

$$\mathbf{E}_L = \mathbf{E}_{L,x} + i\mathbf{E}_{L,y} = \mathbf{E}_0 \exp i(\omega_L t - k_L z - \varphi_0) \; .$$

We now calculate the change of the electron's energy in the combined presence of the undulator and the electromagnetic field. It is well known that a charged particle does not gain energy in any magnetic field, since the Lorentz force is always perpendicular to the particle's velocity. Thus we have to consider only the electric field. As the electromagnetic wave has nothing but electric field components perpendicular to the mean electron beam (z) direction, we now recognize the important role of the undulator field: It generates velocity components of the electrons in the direction of the electric field vector, i.e. in the x and y direction making energy transfer between the electromagnetic wave and the electron beam possible. The electron's energy E is changed at a rate

$$\begin{aligned} \frac{dE}{dt} &= mc^2\frac{d\gamma}{dt} = \vec{v}\cdot\vec{F} = q\cdot\vec{v}\cdot\vec{\mathbf{E}}_L = q\left(\dot{x}\mathbf{E}_{L,x} + \dot{y}\mathbf{E}_{L,y}\right) = q\Re\left(\dot{w}\mathbf{E}_L^*\right) = \\ &= -qc\frac{K\mathbf{E}_0}{\gamma}\sin\{(k_u + k_L)z - \omega_L t + \varphi_0\} = -qc\frac{K\mathbf{E}_0}{\gamma}\sin\Psi \; . \end{aligned}$$

We have used Eq. (7) and the 'ponderomotive phase' defined as $\Psi = (k_u + k_L)z - \omega_L t + \varphi_0$. If we use $z = v_z t = \beta_z ct$, we can write

$$\boxed{\Psi = (k_u + k_L)z - \frac{\omega_L z}{\beta_z c} + \varphi_0} \tag{10}$$

and

$$\boxed{\frac{dE}{dz} = -\frac{q\mathbf{E}_0 K}{\gamma\beta_z}\sin\Psi} \; . \tag{11}$$

The energy dE is taken from or transferred to the radiation field. For most frequencies, dE/dt oscillates very rapidly. A significant energy transfer will only be accumulated if the phase difference between particle motion and electromagnetic wave stays constant with time. Thus, there is a resonance condition give by

$$\Psi = \text{const.} \;\rightarrow\; \frac{d\Psi}{dz} = (k_u + k_L) - \frac{\omega_L}{\beta_z c} = 0 \, .$$

Using $\omega_L = \mathrm{c}k_L$ yields

$$k_u + k_L - \frac{k_L}{\beta_z} = 0 \ .$$

Solving for $\lambda_L = \dfrac{2\pi}{k_L}$ we get the resonance condition

$$\lambda_L = \lambda_u \frac{1-\beta_z}{\beta_z} \approx \lambda_u \left(1-\beta_z\right) \approx \frac{\lambda_u}{2\gamma^2}\left(1+K^2\right) \ . \tag{12}$$

It is important to realize that the resonant wavelength λ_L is identical to the on-axis, first harmonic wavelength spontaneously radiated by the undulator.
With Eq. (12) we have achieved a condition for continuous energy transfer from the electron beam to the electromagnetic wave. However, even if all electrons had exactly the right energy to fulfil this condition when they entered the undulator, they would leave the resonance energy quickly because of the energy transfer to (or from) the wave. Thus, we need to investigate what happens to electrons with energies slightly off resonance. For particles slightly off resonance, the phase Ψ will slip. In order to understand by how much, we note that in Eq. (10) only $\beta_z \approx 1 - 1/2\gamma^2\left(1+K^2\right)$ depends on energy. Writing $\gamma = \gamma_{\text{res}} + \Delta\gamma$ we get

$$\begin{aligned} \frac{d\Psi}{dz} &= \left(k_u + k_L\right) - \frac{\omega_L}{c\left(1 - \dfrac{1+K^2}{2\left(\gamma_{\text{res}}+\Delta\gamma\right)^2}\right)} \approx k_u + k_L - \frac{\omega_L}{\beta_z\left(\gamma_{res}\right)\cdot c} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_{\text{res}}{}^3}\Delta\gamma = \\ &= \frac{\omega_L}{c}\frac{1+K^2}{\gamma_{\text{res}}{}^3}\Delta\gamma = k_u \frac{2}{\gamma_{\text{res}}}\Delta\gamma \ . \end{aligned} \tag{13}$$

Deriving once more with respect to *z* yields $\dfrac{d^2\Psi}{dz^2} = k_u \dfrac{2}{\gamma_{\text{res}}}\dfrac{d\gamma}{dz}$.

Using Eq. (11) in the form $\dfrac{d\gamma}{dz} = -\dfrac{q\mathbf{E}_0 K}{m_0 c^2 \gamma\beta_z}\sin\Psi$ we finally get

$$\boxed{\frac{d^2\Psi}{dz^2} = -\frac{2q}{m_0c^2}\frac{\mathbf{E}_0 Kk_u}{\gamma_{res}^2\beta_z}\sin\Psi = -\Omega^2\sin\Psi \ \text{ with } \ \Omega^2 = \frac{2q}{m_0c^2}\frac{\mathbf{E}_0 Kk_u}{\gamma_{res}^2\beta_z}} \ . \tag{14}$$

This is pendulum equation in the $\Delta\gamma - \Psi$ phase space: electrons with little deviation from resonance energy or from synchronous phase perform periodic oscillations, see Fig. 2. This is equivalent to the synchrotron oscillations in storage rings, except that the 'bucket' length is now the optical wavelength. Similarly, in synchrotron oscillation, particles within the separatrix get bunched.

The energy lost (or gained) by an electron increases (or decreases) the field energy. Thus, as seen from Eq. (11) and illustrated in Fig. 3, there is gain or loss in field energy per undulator passage depending on where the electron starts in the $\Delta\gamma - \Psi$ phase space.

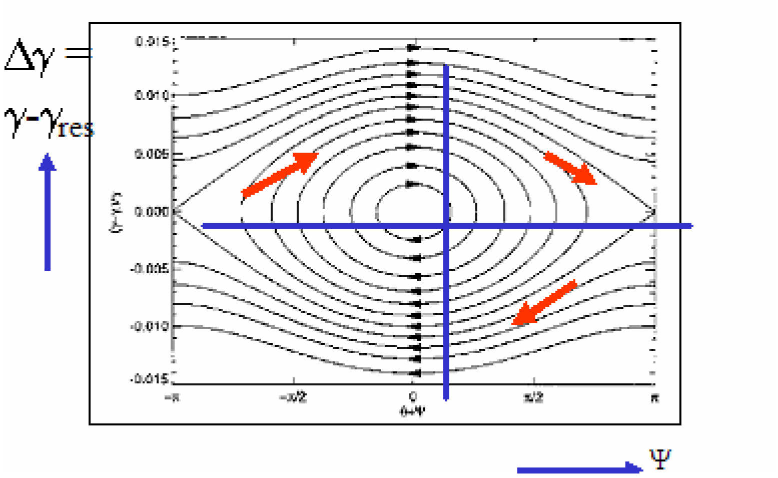

Fig. 3: In the presence of the undulator and electromagnetic field, buckets are formed where electrons perform periodic oscillation if the deviation from resonance energy and from synchronous phase is small. In contrast to synchrotron oscillation buckets, the longitudinal size of our buckets is very small, i.e., the optical wavelength.

2.3.1 The separatrix

In order to determine the parameters of the separatrix, we look for a first integral of Eq. (14):

Multiplying $\Psi'' = -\Omega^2 \sin\Psi$ by $2\Psi'$ on both sides and using $2\Psi'\Psi'' = \frac{d}{dz}(\Psi')^2$ yields

$$\int 2\Psi'\Psi'' dz = (\Psi')^2 = 2\int \Omega^2 \sin\Psi \Psi' dz = 2\int -\Omega^2 \sin\Psi d\Psi = 2\Omega^2 \cos\Psi + \text{const.} \; .$$

With $\Psi' = k_u \frac{2}{\gamma_{res}} \Delta\gamma$ this reads $\left(k_u \frac{2}{\gamma_{res}} \Delta\gamma\right)^2 = 2\Omega^2 \cos\Psi + \text{const.}$, thus

$$\boxed{(\Delta\gamma)^2 - \frac{q \mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \cos\Psi = \text{const.}} \; , \tag{15}$$

with const. determined by initial conditions.

There are two cases to be distinguished:

Case 1: $\text{const.} < \frac{q \mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}$. Then, $\Delta\gamma = \sqrt{\text{const.} + \frac{q \mathbf{E}_0 K}{m_0 c^2 k_u \beta_z} \cos\Psi}$ has real solutions only within a limited range of phases. This is the case of rotation within the separatrix.

Case 2: $\text{const.} > \frac{q \mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}$. In this case all phases are possible, but $\Delta\gamma = 0$ cannot be reached. As a consequence, the electron performs 'libration' outside the separatrix. The separatrix is defined by the limiting case: $\text{const.} = \frac{q \mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}$. Thus, the separatrix is defined by the equation

$$\boxed{(\Delta\gamma)^2 = \frac{q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}(1+\cos\Psi)} \quad . \tag{16}$$

The height of the separatrix is given by: $\Delta\gamma_{\max} - \Delta\gamma_{\min} = \sqrt{\dfrac{2q\mathbf{E}_0 K}{m_0 c^2 k_u \beta_z}}$, (17)

i.e. it is determined by the strengths of both the external electromagnetic wave $\mathbf{E}_0$ and of the undulator field (through K).

2.3.2 Power gain

In practice, an electron beam consists of many particles distributed smoothly throughout the phases, it is unclear from the previous analysis whether a significant overall amplification of the electromagnetic wave can take place. To determine the 'power gain' of the FEL in the presence of the entire beam, our most important assumption is that the amplitude of the electromagnetic wave changes little during one passage of the electron beam: i.e. the power gain (as defined below) is much smaller than unity: |G| < 1. This is the 'low-gain approximation', the subject of this section. We also assume an initially monoenergetic beam with some deviation $\Delta\gamma$ from resonance energy. Let us define the power gain G_i due to a particle identified by the index *i* by

$$\begin{aligned} G_i &= \frac{\text{gain of field energy produced by electron } i}{\text{total field energy}} = \frac{-mc^2\left(\gamma_i(z=L_u)-\gamma_i(0)\right)}{\frac{\varepsilon_0}{2}\mathbf{E}_0^2 \cdot V} \\ &= \frac{-mc^2\gamma_{\text{res}}\left(\Psi_i'(z=L_u)-\Psi_i'(0)\right)}{\varepsilon_0\mathbf{E}_0^2 \cdot Vk_u} \quad \left[\text{using Eq. (13) } \Delta\gamma = \frac{\gamma_{\text{res}}}{2k_u}\Psi'\right] . \end{aligned} \tag{18}$$

The undulator length is L_u. Calculation of G_i requires solution of the pendulum equation (14) $\Psi'' = -\Omega^2 \sin\Psi$ for $\Psi(z)$. This is done iteratively. Start with the ansatz

$$\Psi(z) = \Psi_0 + \Psi_0' \cdot z + \delta\Psi(z) \ , \tag{19}$$

where $\delta\Psi(z)$ is the higher order term.

Step 1: $\delta\Psi(z) = 0$.

Using the ansatz, a first integral of Eq. (14) is then:

$$\Psi'^{(1)} = \Psi_0' - \Omega^2 \int_0^z \sin(\Psi_0 + \Psi_0' \cdot \tilde{z})d\tilde{z} = \Psi_0' - \Omega^2 \frac{1}{\Psi_0'}\left[\cos\Psi_0 - \cos(\Psi_0 + \Psi_0' \cdot z)\right] \tag{20}$$

The gain of the entire beam (consisting of N_p particles) is given by

$$G = \sum_i G_i = \langle G_i \rangle_{\Psi_{i0}} \cdot N_p \tag{21}$$

Averaging with respect to the initial phases is denoted by $\langle \ \rangle_{\Psi_{i0}}$ and yields

$$\left\langle \Psi'^{(1)} - \Psi'_0 \right\rangle_{\Psi_0} = \left\langle -\Omega^2 \frac{1}{\Psi'_0} \left[\cos\Psi_0 - \cos\left(\Psi_0 + \Psi'_0 \cdot z\right) \right] \right\rangle_{\Psi_0} = 0 \ . \tag{22}$$

The important result is that, in first order, the average gain G is zero!

Step 2: $\delta\Psi(z) \neq 0$, calculating $\delta\Psi(z)$ using the results of step 1, Eq. (20).

$$\begin{aligned} \delta\Psi(z) &= -\Omega^2 \frac{1}{\Psi'_0} \int_0^z \left[\cos\Psi_0 - \cos\left(\Psi_0 + \Psi'_0 \cdot \tilde{z}\right) \right] d\tilde{z} = \\ &= -\Omega^2 \frac{1}{\Psi'_0} \left[z \cdot \cos\Psi_0 - \frac{1}{\Psi'_0} \sin\left(\Psi_0 + \Psi'_0 \cdot z\right) + \frac{1}{\Psi'_0} \sin\Psi_0 \right] . \end{aligned} \tag{23}$$

Using the ansatz again, the first integral of Eq. (14) now reads

$$\Psi'^{(2)} - \Psi'_0 = -\Omega^2 \int_0^z \sin\left(\Psi_0 + \Psi'_0 \cdot \tilde{z} + \delta\Psi(\tilde{z})\right) d\tilde{z} \approx$$

$$\approx -\Omega^2 \int_0^z \left[\sin\left(\Psi_0 + \Psi'_0 \cdot \tilde{z}\right) + \cos\left(\Psi_0 + \Psi'_0 \cdot \tilde{z}\right) \delta\Psi(\tilde{z}) \right] d\tilde{z} \ .$$

The approximation is valid for $\delta\Psi(\tilde{z}) \ll \pi$, keep this in mind when using the results. Plugging in $\delta\Psi(\tilde{z})$ and averaging with respect to phases yields

$$\left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} = \Omega^4 \frac{1}{\Psi'^2_0} \left\langle \int_0^{Lu} \left[\tilde{z}\Psi'_0 \cos^2\Psi_0 \cdot \cos\left(\Psi'_0 \cdot \tilde{z}\right) - \sin^2\Psi_0 \cdot \sin\left(\Psi'_0 \cdot \tilde{z}\right) \right] d\tilde{z} \right\rangle_{\Psi_0} =$$

$$= \frac{\Omega^4}{2\Psi'^2_0} \int_0^{Lu} \left[\tilde{z}\Psi'_0 \cos\left(\Psi'_0 \cdot \tilde{z}\right) - \sin\left(\Psi'_0 \cdot \tilde{z}\right) \right] d\tilde{z} \ .$$

Here we used $\left\langle \cos(\alpha+\beta)\sin(\alpha+\beta) \right\rangle_\alpha = 0$ and $\left\langle \cos^2\alpha \right\rangle_\alpha = \frac{1}{2}$; and $\left\langle \sin^2\alpha \right\rangle_\alpha = \frac{1}{2}$.

By partial integration as follows

$$\int_0^{L_u} \underbrace{\tilde{z}\Psi'_0}_{u} \underbrace{\cos}_{v'}(\Psi'_0\tilde{z}) d\tilde{z} = \underbrace{\tilde{z}\Psi'_0}_{u} \underbrace{\frac{1}{\Psi'_0} \sin(\tilde{z}\Psi'_0) \Big|_0^{L_u}}_{v} - \int_0^{L_u} \underbrace{\Psi'_0}_{u'} \underbrace{\frac{1}{\Psi'_0} \sin(\Psi'_0\tilde{z})}_{v} d\tilde{z} \ ,$$

we get $\left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} = \dfrac{\Omega^4}{2\Psi_0'^2}\left[L_u \sin(\Psi'_0 L_u) - 2\int_0^{Lu} \sin(\Psi'_0 \cdot \tilde{z})\, d\tilde{z} \right] =$

$$= \frac{\Omega^4}{2\Psi_0'^2}\left[L_u \sin(\Psi'_0 L_u) + 2\frac{1}{\Psi'_0}\left(\cos(\Psi'_0 L_u) - 1\right) \right] =$$

(using $\sin 2x = 2\sin x \cos x$ and $\cos 2x - 1 = -2\sin^2 x$)

$$= \Omega^4 \left(\frac{1}{\Psi_0'^2} L_u \sin\frac{\Psi'_0 L_u}{2} \cos\frac{\Psi'_0 L_u}{2} - \frac{2}{\Psi_0'^3}\sin^2\frac{\Psi'_0 L_u}{2} \right).$$

The latter expression can also be put in the form $\left\langle \Psi'^{(2)} - \Psi'_0 \right\rangle_{\Psi_0} = \Omega^4 \dfrac{\mathrm{d}}{\mathrm{d}\Psi'_0}\left\{ \dfrac{\sin^2 \frac{\Psi'_0 L_u}{2}}{\Psi_0'^2} \right\}.$

With Eqs. (18) and (21), the gain can be written

$$G = N_p \frac{-mc^2\gamma_{\text{res}}\left(\Psi'_i(z = L_u) - \Psi'_i(0)\right)}{\varepsilon_0 \mathbf{E}_0^2 \cdot V k_u} = -\frac{mc^2\gamma_{res} N_p}{\varepsilon_0 \mathbf{E}_0^2 \cdot V k_u}\Omega^4 \frac{\mathrm{d}}{\mathrm{d}\Psi'_0}\left\{ \frac{\sin^2 \frac{\Psi'_0 \cdot L_u}{2}}{\Psi_0'^2} \right\}.$$

Using the abbreviations $\xi = \dfrac{\Psi'_0 \cdot L_u}{2}$, $n_p = \dfrac{N_p}{V}$, $L_u = N_u \lambda_u$ (with N_u the number of undulator periods) and get the final result [1]

$$\boxed{G = -\frac{\pi q^2 N_u^3 \lambda_u^2 K^2 n_p}{\varepsilon_0 mc^2 \gamma^3} \frac{\mathrm{d}}{\mathrm{d}\xi}\left\{ \frac{\sin^2 \xi}{\xi^2} \right\}}. \tag{24}$$

The functions $\dfrac{\sin^2 \xi}{\xi^2}$ and $\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left\{ \dfrac{\sin^2 \xi}{\xi^2} \right\}$ are plotted in Fig. 4.

To summarize, the key assumptions based on this result are a helical undulator, perfect overlap of electron and radiation field, perfect electron beam (zero emittance and no momentum spread). Note that the approximations do not assume that the beam is located inside the separatrix, i.e. the external electromagnetic field may be so weak that the separatrix covers only a small fraction of the gain curve.

For the interpretation of Eq. (24) it is useful to express ξ in the form

$$\xi = \frac{\Psi'_0 \cdot L_u}{2} = \frac{k_u L_u \Delta\gamma}{\gamma_{res}} = 2\pi N_u \frac{\Delta\gamma}{\gamma_{res}} \tag{25}$$

[1] Using the same volume V in the definition of the particle density as in the expression for the total field energy [see Eq. (18)] means that we assume perfect overlap of electron beam and electromagnetic field.

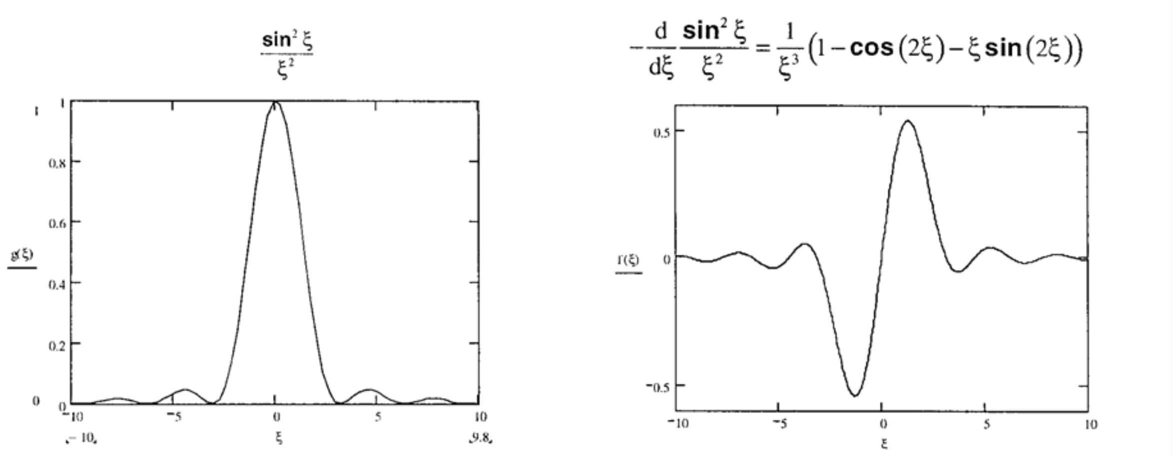

Fig. 4: In low-gain approximation, the dependency of power gain on the initial momentum (right) can be written as the derivative of the line-shape function (left) of the spontaneous undulator radiation

Thus, we can interpret the result as follows: There is no net gain ($G = 0$), to first order in the iteration, because phase-space motion is (almost) symmetric. In other words, the same amount of particles move up as move down. In second order, however, for positive $\Delta\gamma$, the particle motion with positive phase goes downwards more rapidly than the motion of the others goes upwards, i.e. there is positive gain if the electron energy is slightly above resonance energy ($\Delta\gamma = 0$). This is illustrated in Fig. 5. There is no gain for particles precisely on resonance energy ($\Delta\gamma = 0$). For $\Delta\gamma < 0$, the gain is negative, this means the beam extracts energy from the electromagnetic wave, and is accelerated. A device accelerating electrons by an electromagnetic wave using this mechanism is called an 'inverse FEL'.

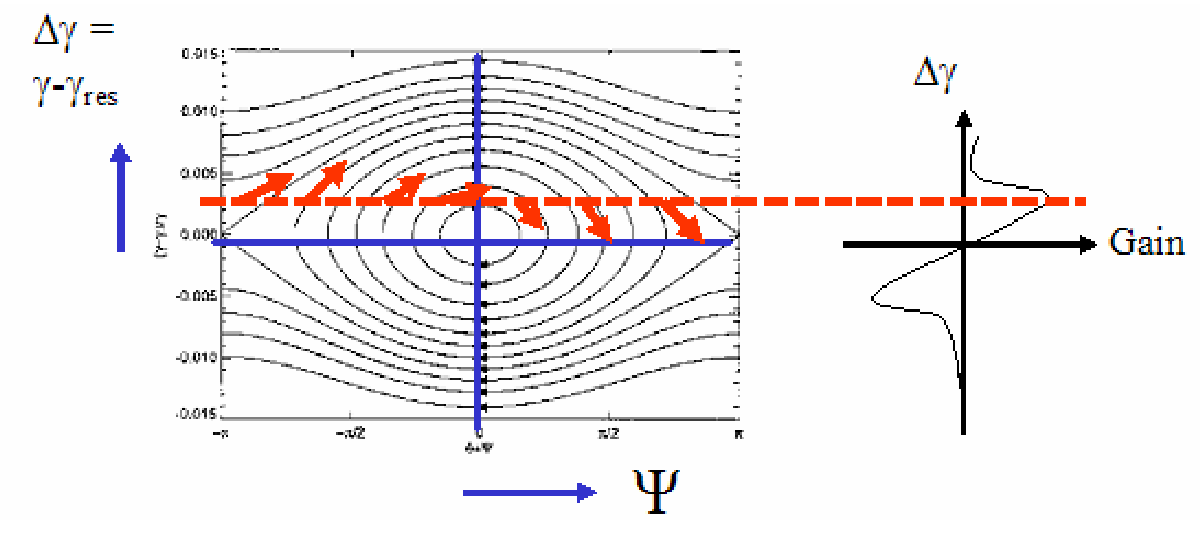

Fig. 5: If the electron energy is slightly above resonance energy (red broken line), some particles lose, while some gain energy. On average, electron energy is pumped into the electromagnetic wave (positive power gain), this is a second-order effect.

2.3.3 Madey theorem

Another interpretation of Eq. (24) uses the relation [see Eq. (12)] $\Delta\omega/2\omega_{\text{res}}=\Delta\gamma/\gamma_{res}$ connecting $\Delta\gamma$ to the deviation of the angular frequency $\Delta\omega$ from its resonance value ω_{res}. Using this relation, we can write

$$G = -\frac{4\pi q^2 N_u^2 \lambda_u n_p}{\varepsilon_0 mc\gamma} \frac{K^2}{(1+K^2)} \frac{d}{d\omega} \frac{\sin^2\left(\pi N_u \dfrac{\Delta\omega}{\omega_{\text{res}}}\right)}{\left(\pi N_u \dfrac{\Delta\omega}{\omega_{\text{res}}}\right)^2} .$$

The expression

$$\frac{\sin^2\left(\pi N_u \dfrac{\Delta\omega}{\omega_{\text{res}}}\right)}{\left(\pi N_u \dfrac{\Delta\omega}{\omega_{\text{res}}}\right)^2}$$

is just the spectral line-shape function of spontaneous radiation of an undulator with N_u periods in the vicinity of the first harmonic resonance frequency ω_{res}, see the left-hand side of Fig. 4. The following statement is called the Madey-theorem [5]:

The gain function of low-gain FEL emission is the derivative of the line-shape function of spontaneous undulator radiation**.**

2.3.4 The optical cavity

The amount of radiation energy produced per undulator passage is $\Delta E = G \cdot E_i$, where E_i represents the radiation energy before the electron bunch passes the undulator. One might think that a power gain of a few per cent (i.e. a low-gain FEL) is of no interest. However, it is important to realize that the gain is independent of the strength of the initial, external electromagnetic field, for example whatever the initial field, it will be amplified by this gain factor. The radiation produced can be accumulated successively with a pair of mirrors arranged to form an optical cavity as shown in Fig. 6, if on each round trip of radiation a fresh electron bunch is available. After N round trips, the total power gain is $G_{\text{total}} = G^N$, which may be a very large number, even if G is not much larger than unity.

At the end of this process, there is a great deal of radiation energy stored in the optical cavity, so that amplification of this energy by even a few per cent is a large quantity in terms of absolute numbers. In other words, the electrons are stimulated to emit radiation because of the presence of the existing field. In fact, the amplification process in the FEL can be described quantum-mechanically in terms of emission and absorption of radiation quanta (photons), and with the properties of FEL radiation like coherence and many photons per coherence volume — it justifies the notion of a ‘laser’. Many of the early FEL papers were written in a quantum mechanics context which explains the generous use of quantum-based terminology in the FEL community. Nevertheless, the description of FELs in terms of classical physics is, with a few rare exceptions, perfectly correct. The FEL is a ‘classical device’.

A fraction of the radiation energy gained is extracted through one of the semi-transparent mirrors. Of course, the mirror transparency must be arranged so that the total power losses of the optical cavity by extraction or absorption do not exceed the power gain.

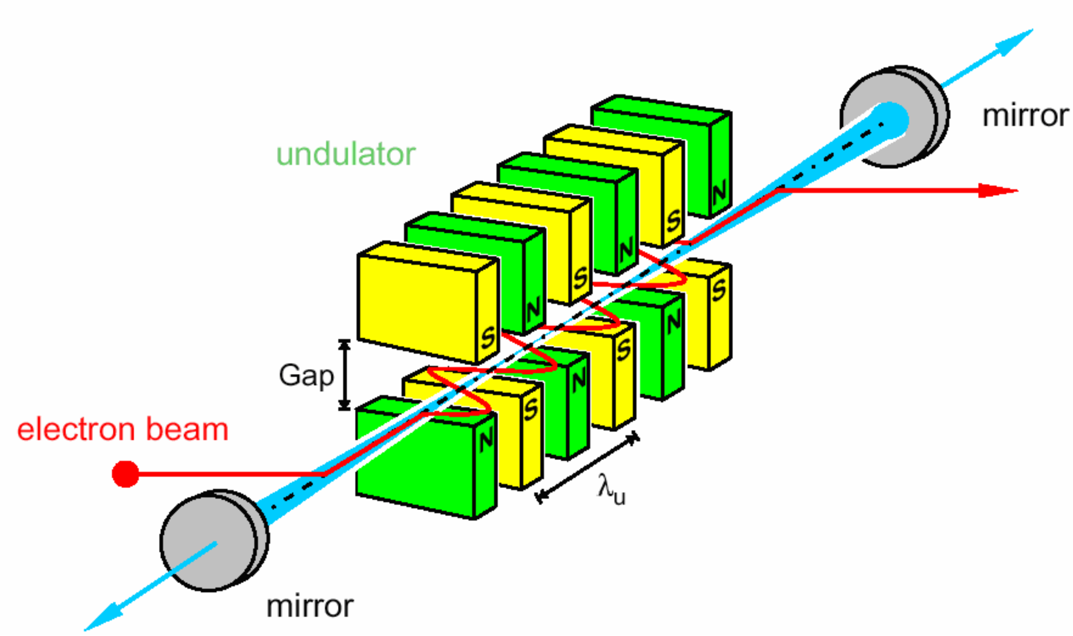

Fig. 6: A free-electron laser in oscillator mode (R. Bakker). Large radiation power can be generated if the radiation field can be stored in an optical cavity, and if many electron bunches pass the cavity with timing synchronized to the round-trip of radiation within the oscillator: even if there is only a few per cent field gain per passage of an electron bunch (low-gain FEL). If the power gain exceeds the accumulated mirror losses (including the semi-transparent mirror for extraction), the stored power increases passage-by-passage in an exponential way until saturation is reached.

2.3.5 Saturation

If the change of electron energy within one undulator passage becomes comparable with $\gamma_{res}/2\pi N_u$, the phase advance per undulator passage becomes large according to Eq. (13), so that the assumption $\delta\Psi(\tilde{z}) \ll \pi$ made for our gain calculation is violated. Also, the assumption of a quasi-monoenergetic electron beam is violated, i.e. the parameter ξ defined in Eq. (25) varies significantly during the undulator passage, depending on the longitudinal position within the separatrix. Thus, Eq. (24) becomes a useless calculation tool for overall gain because, based on violated assumptions, it is now inaccurate. In this case, the gain must be calculated numerically.

According to Eq. (17), the height of the separatrix grows with the radiation energy stored in the cavity. More and more electrons get trapped within the separatrix, providing an efficient mechanism for longitudinal bunching of the electron beam at the optical wavelength (an effect called ‘micro-bunching’, to be distinguished from longitudinal bunching of the entire electron bunch). The gain process saturates if most of the electrons get micro-bunched, i.e. if the electron density is almost completely modulated at the resonance wavelength. According to Eq. (4) and the explanations thereof, the undulator radiation power exceeds spontaneous radiation power by a large factor *N* comparable to the number of electrons per resonance wavelength.

This section only solves the kinematic problem of an electron beam in the combined presence of a given electromagnetic wave and an undulator field where the gain of radiation power was derived from an energy conservation argument. The kinetic energy taken from the electron must go into the radiation energy — where else? It is obvious that, for a more thorough analysis, one has to solve both the kinematic problem and the electrodynamical problem (i.e. the generation of radiation by a modulated charge according to Maxwell’s equations) simultaneously. Such a detailed analysis of the

FEL oscillator, including the analysis of saturation, can be found in the literature, [2]. The next section presents this kind of treatment for the high-gain, single-pass FEL. Indeed, the low-gain results in this section can be derived from the general result as a special case, low-gain approximation, which with hindsight justifies the low-gain treatment given above.

Although storage-ring FELs are beyond the scope of this article, we conclude this section with a remark on FEL oscillators driven by electron bunches in a storage ring. This kind of arrangement is very attractive, since the electron bunch can be used many times (i.e. once per revolution), and reliable operation can be expected on account of the inherent stability of storage rings in terms of timing and bunch population. However, there are inherent drawbacks: As the same electron bunch is used many times, the electron-beam dynamics in the storage ring must be taken into account. As described above, the energy width of the electron beam is considerably increased close to saturation at each passage of the FEL. This energy broadening accumulates from turn-to-turn. It is somewhat compensated by radiation damping, but the saturation process remains drastically determined by this effect [6]. Even if one considered using the beam of a storage ring only once per damping time, there is a fundamental issue making such an electron bunch unattractive in some cases: The product of bunch length and energy width (longitudinal emittance) is determined by quantum fluctuation effects in a storage ring and cannot be made as small as in a linear accelerator.

2.3.6 Start-up from noise

In order to achieve maximum gain, ξ should be ~ +1, i.e. $\Delta\gamma \approx + \gamma/2\pi N_u$. The electron-beam energy should be above resonance energy by that amount. This is easy to achieve if the initial electromagnetic field is provided by an external source. The external wavelength determines, together with the undulator parameters, the resonance energy γ_{res} of the electron beam. Therefore, we simply have to set the electron-beam energy to $\gamma_{res} + \Delta\gamma$.

What happens if there is no external radiation source? The FEL can still work, if the spontaneous radiation of the undulator is used. With two mirrors, this radiation has to be reflected back to the entrance of the undulator, and it must be synchronized longitudinally with the next electron bunch for overlap in the undulator (see Fig. 6).

Unfortunately, if we want to use the centre of the spectrum of the spontaneous undulator radiation as the ‘external wave’, there will be no FEL gain, because this wavelength, and the beam energy, fulfill the resonance condition exactly. It is helpful that the spontaneous spectrum has the same width [see Eq. (24) and Fig. 4] as the gain curve, therefore there is always significant power at a wavelength with high gain.

This can happen in two ways:

1. If the bandwidth of the optical resonator formed by the two mirrors is very large, then the FEL will ‘automatically’ amplify only that part of the spectrum with positive gain. This will happen with the lower frequency part of the spontaneous spectrum. For this wavelength to be resonant, the beam energy would have to be smaller than it actually is, so the actual beam energy will be slightly above resonance, as it should be for positive gain.
2. If the bandwidth of the optical resonator is small (normal case), it should be tuned below the centre frequency of the spontaneous spectrum (same argument as before). This is illustrated in Fig. 7.

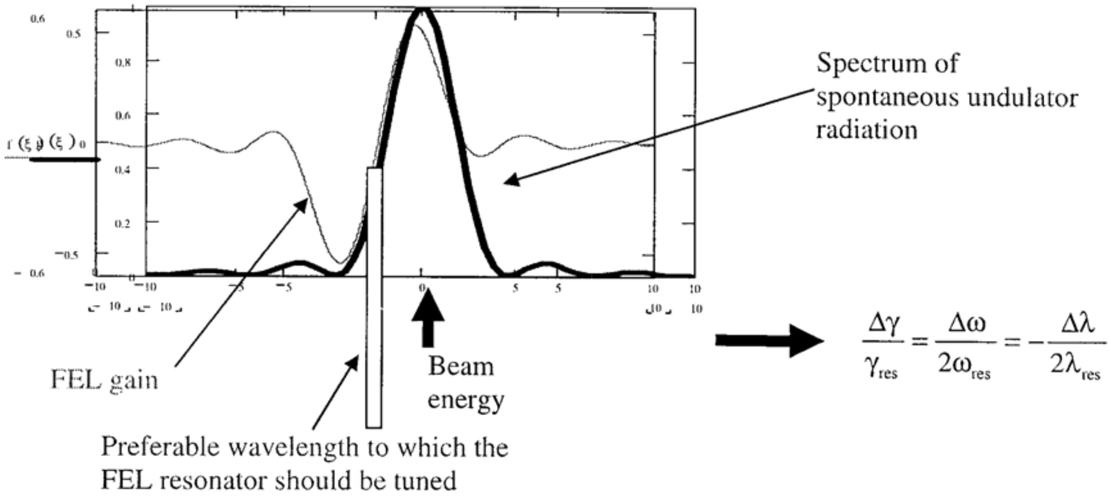

Fig. 7: When starting from noise, the spontaneous radiation of the undulator serves as an 'external electromagnetic wave' to be amplified in the FEL. For optimum gain, the electron energy should be chosen so that it samples the gain curve where it is positive and maximum. Since the gain curve is centred with respect to the resonance energy (rather than w.r.t. the beam energy), this condition can be met if the resonant wavelength of the optical cavity (yellow line) is tuned below the electron beam energy. Then, only the low-frequency wing of the undulator spectrum gets amplified, but it is guaranteed by the strict relation between the width of the spectrum and the width of the gain curve (see section 2.3.3) that there is sufficient radiation power in this portion of the spectrum to serve as input signal.

3 The high-gain free-electron laser

If the radiation power gained within a single passage of the electron beam through the undulator is comparable or much larger than the input radiation power, the low-gain approximation is not applicable any more. In this case, we have to take into account the time-dependence of the increasing electromagnetic wave as determined by the motion of the electrons in the beam. On the other hand, just this motion is determined by the electromagnetic field amplitude at any point in time and space. Thus, we need to treat the evolution of electron kinematics and electromagnetic field amplitude in a self-consistent manner.

The purpose of this section is to derive the key equations from first principles, motivating the approximations and providing some realistic numbers for illustration. The treatment follows closely the one given in Ref. [2].

3.1 Generation of electromagnetic fields by the electron beam

We start with a derivation of the wave equation for the electric field from Maxwell's equations. From Maxwell's equation $\mathrm{rot}\,\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$ we get $\mathrm{rot}\,\mathrm{rot}\,\vec{\mathbf{E}} = \mathrm{grad}\,\mathrm{div}\,\vec{\mathbf{E}} - \nabla^2\,\vec{\mathbf{E}} = -\mu_0 \mathrm{rot}\frac{\partial \vec{\mathbf{H}}}{\partial t}$. (26)

Next, we derive Maxwell's equation $\mathrm{rot}\,\vec{\mathbf{H}} = \vec{j} + \varepsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$ once more with respect to time and get

$$\mathrm{rot}\frac{\partial \vec{\mathbf{H}}}{\partial t} = \frac{\partial \vec{j}}{\partial t} + \varepsilon_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} .$$

If we further use Maxwell's equations $\varepsilon_0 \mathrm{div}\,\vec{\mathbf{E}} = \rho$, Eq. (26) can be written in the form

$$\frac{1}{\varepsilon_0} grad\rho - \nabla^2 \vec{\mathbf{E}} = -\mu_0 \frac{\partial \vec{j}}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \ .$$

Using $\mu_0 \varepsilon_0 = \frac{1}{c^2}$, this reads

$$\left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2} \right) \vec{\mathbf{E}} = \mu_0 \frac{\partial \vec{j}}{\partial t} + \frac{1}{\varepsilon_0} \nabla \rho \,, \tag{27}$$

which is the well-known wave equation for the electric field. In the following approximation, we restrict ourselves to a one-dimensional treatment of the FEL, i.e. we consider a purely transverse electromagnetic field. This means that we neglect diffraction effects, which is certainly questionable for long wavelengths. With this approximation, Eq. (27) reads

$$\frac{\partial^2 \mathbf{E}_\perp}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_\perp}{\partial t^2} = \mu_0 \frac{\partial j_\perp}{\partial t} + \frac{1}{\varepsilon_0} \nabla_\perp \rho \,, \tag{28}$$

with the index $\perp$ denoting the vector component perpendicular to the direction of electron propagation. The term $1/\varepsilon_0 \nabla_\perp \rho$ in Eq. (28) can be neglected because its contribution to radiation generation is small in all practical cases.[2] The transverse electric field $\mathbf{E}_\perp$ of the electromagnetic wave can be written in the form

$$\vec{\mathbf{E}}_\perp = \mathbf{E}_0 \begin{pmatrix} \cos(\omega_L t - k_L z - \varphi_0) \\ \sin(\omega_L t - k_L z - \varphi_0) \\ 0 \end{pmatrix} .$$

The magnetic field of the electromagnetic wave is then $\vec{\mathbf{B}}_\perp = \frac{1}{c\omega_L} \dot{\vec{\mathbf{E}}}_\perp$.

In analogy to the previous section, we take advantage of the fact that we have to deal with only two components of $\vec{\mathbf{E}}$ and define a complex electric field given by $\mathbf{E} = \mathbf{E}_{\perp,x} + i\mathbf{E}_{\perp,y} = \mathbf{E}_0 \exp i(\omega_L t - k_L z - \varphi_0)$. The only difference from the low-gain case is now that the amplitude $\mathbf{E}_0$ and the phase φ_0 (which we will call ψ_E now) may vary with z (though slowly compared to $\omega_L t$). We thus separate the slow part from the rapidly oscillating part by writing: $\mathbf{E} = \mathbf{E}_0(z) \exp i(\omega_L t - k_L z - \psi_E) = \tilde{\mathbf{E}}_0^*(z) \exp i(\omega_L t - k_L z)$, with the slow part $\tilde{\mathbf{E}}_0(z) = \mathbf{E}_0(z) \exp i\psi_E$ and $\tilde{\mathbf{E}}_0^*(z)$ its complex conjugate (c.c.). Equation (28), re-written for $\mathbf{E}_{\perp,x} + i\mathbf{E}_{\perp,y} = \tilde{\mathbf{E}}_0^*(z) \exp i(\omega_L t - k_L z)$ reads

[2] For a more detailed justification, see Ref. [2], section 4.1.

$$
\begin{aligned}
&\frac{\partial^2(\mathbf{E}_{\perp,x}+i\mathbf{E}_{\perp,y})}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2(\mathbf{E}_{\perp,x}+i\mathbf{E}_{\perp,y})}{\partial t^2}=\\
&=\frac{\partial^2}{\partial z^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,i(\omega_L t-k_L z)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,i(\omega_L t-k_L z)=\\
&\frac{\partial}{\partial z}\left[\tilde{\mathbf{E}}_0^*(z)\frac{\partial}{\partial z}\mathbf{exp}\,i(\omega_L t-k_L z)+\mathbf{exp}\,i(\omega_L t-k_L z)\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]-\ldots\\
&\ldots\frac{1}{c^2}\frac{\partial}{\partial t}\left[\tilde{\mathbf{E}}_0^*(z)\frac{\partial}{\partial t}\mathbf{exp}\,i(\omega_L t-k_L z)+\mathbf{exp}\,i(\omega_L t-k_L z)\cancel{\frac{\partial}{\partial t}\tilde{\mathbf{E}}_0^*(z)}\right]=\\
&\frac{\partial}{\partial z}(\tilde{\mathbf{E}}_0^*(z))\cdot(-ik_L)\,\mathbf{exp}\,i(\omega_L t-k_L z)+\tilde{\mathbf{E}}_0^*(z)(-k_L^2)\,\mathbf{exp}\,i(\omega_L t-k_L z)+\ldots\\
&\ldots(-ik_L)\,\mathbf{exp}\,i(\omega_L t-k_L z)\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)+\mathbf{exp}\,i(\omega_L t-k_L z)\frac{\partial^2}{\partial z^2}\tilde{\mathbf{E}}_0^*(z)-\ldots\\
&\ldots\cancel{\frac{1}{c^2}\frac{\partial}{\partial t}(\tilde{\mathbf{E}}_0^*(z))\cdot i\omega_L\,\mathbf{exp}\,i(\omega_L t-k_L z)}-\frac{1}{c^2}\tilde{\mathbf{E}}_0^*(z)(-\omega_L^2)\,\mathbf{exp}\,i(\omega_L t-k_L z)+0
\end{aligned}
$$

where we have made use of our assumption that the complex field amplitude does depend on the longitudinal coordinate z, but not (explicitly) on time t, i.e. $\frac{\partial}{\partial t}\tilde{\mathbf{E}}_0^*(z)=0$. We further neglect the second derivative of the field amplitude with respect to z because it is assumed to vary only slowly and get:

$$
\begin{aligned}
&\frac{\partial^2(\mathbf{E}_{\perp,x}+\mathrm{i}\mathbf{E}_{\perp,y})}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2(\mathbf{E}_{\perp,x}+\mathrm{i}\mathbf{E}_{\perp,y})}{\partial t^2}=\\
&=-\left[2\mathrm{i}k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)-k_L^2\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)-\frac{(-\omega_L^2)}{c^2}\tilde{\mathbf{E}}_0^*(z)\,\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)=\\
&=-\left[2\mathrm{i}k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]\mathbf{exp}\,\mathrm{i}(\omega_L t-k_L z)=\mu_0\frac{\partial(j_{\perp,x}+\mathrm{i}\cdot j_{\perp,y})}{\partial t}=\ \mathrm{i}\mu_0\frac{\mathrm{K}}{\gamma}\mathbf{exp}(\mathrm{i}k_u z)\ \frac{\partial j_z}{\partial t}\ .
\end{aligned}
\tag{29}
$$

Here we made use of $\omega_L/c=k_L$. Also, we were able to relate the transverse components of the current density to its longitudinal component, since we know from Eq. (7) how electrons move in the presence of the helical undulator: Namely, because of $\vec{j}=r\vec{v}$, we were able to write $(\mathrm{j}_\mathrm{x}+i\cdot j_y)=(v_\mathrm{x}+i\cdot v_y)\frac{j_z}{v_z}=$ [see Eq.(7)] $=\mathrm{i}\frac{c}{v_z}\frac{K}{\gamma}e\mathbf{xp}(ik_u z)j_z\approx\ \mathrm{i}\frac{K}{\gamma}e\mathbf{xp}(ik_u z)j_z$.

Collecting the rapidly oscillating term and using $\Psi=(k_u+k_L)z-\omega_L t$, Eq. (29) re-writes:

$$
\boxed{-\left[2k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right]=\ \mu_0\frac{\mathrm{K}}{\gamma}\frac{\partial j_z}{\partial t}\mathbf{exp}\,\mathrm{i}(k_u z+k_L z-\omega_L t)=\mu_0\frac{\mathrm{K}}{\gamma}\frac{\partial j_z}{\partial t}\mathbf{exp}\,\mathrm{i}\Psi}\ .
\tag{30}
$$

The message of the equation is pretty simple: The electromagnetic field amplitude is generated by the time-dependent current density.

To proceed further, we have to say something about the current density j_z.

3.2 Kinematics of electrons in phase space

The term j_z is determined by the initial charge distribution and its evolution in the presence of the electromagnetic field and the undulator field. We know that electron dynamics is governed by the Hamiltonian

$$H(p_z, z, t) = \left[\left(p_z c - qA_z \right)^2 + q^2 \left(A_\perp + A_u \right)^2 + m^2 c^4 \right]^{1/2} + q\phi ,$$

with A_u describing the undulator field, and A_z, ϕ the space charge. Applying a canonical transformation, we can change from the canonical pair of coordinates z/p_z to Ψ/γ (actually, the pair is $\omega_\perp/m_0c^2 \cdot \Psi/\gamma$, but $\omega_\perp/m_0c^2$ is constant), with $\Psi = (k_u + k_L)z - \omega_L t$ and $\gamma m_0 c^2$ the kinetic energy of the electron. A consequence of Hamiltonian mechanics is Liouville's theorem, stating that phase space density f along the particle's motion is constant. Phase space motion must be described in any pair of canonically conjugate variables, and we choose Ψ/γ. In coordinates z, γ, Ψ, this theorem reads[3]

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \Psi}\frac{\partial \Psi}{\partial z} + \frac{\partial f}{\partial \gamma}\frac{\partial \gamma}{\partial z} = 0 , \tag{31}$$

which is also called 'Vlasov equation'.

We have seen from Eq. (11) how the electron energy changes in the presence of an electromagnetic field and an undulator field. In addition to Eq. (11), we now include the energy gain due to the presence of a longitudinal space charge field $\mathbf{E}_z$:

$$\frac{d\gamma}{dz} = -\frac{q\mathbf{E}_0 K}{m_0 c^2 \gamma_0 \beta_z} \sin\left(\Psi + \psi_E \right) + \frac{q\mathbf{E}_z}{m_0 c^2} .$$

Of course, we have to understand that the electrical field strength $\mathbf{E}_0$ is no longer constant. Therefore, we also allow for a slowly varying phase of the electromagnetic field, described by ψ_E. Note $d\Psi/dz$ can be determined from Eq. (13):

$$\frac{d\Psi}{dz} = k_u + k_L - \frac{\omega_L}{\beta_z(\gamma_0) \cdot c} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma ,$$

where $\Delta\gamma$ denotes the deviation from γ_0. In general, we allow γ_0 to deviate slightly from resonance energy γ_{res} described by the detuning parameter

$$C(\gamma) = k_u + k_L - \frac{\omega_L}{\beta_z(\gamma) \cdot c} , \quad \text{i.e.} \quad C(\gamma_{res}) = 0 .$$

(Note: You may ask why the deviation from resonance energy is split into two terms, γ_0 and $\Delta\gamma$. The reason will become clear below, where we use $\Delta\gamma$ to describe the energy distribution of the beam

[3] We use the longitudinal coordinate z as independent variable instead of time t, which in fact means another canonical transformation.

around the centre γ_0.) We get:

$$\frac{d\Psi}{dz} = C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\,.$$

Equation (31) now reads

$$\frac{\partial f}{\partial z} + \left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma \right)\frac{\partial f}{\partial \Psi} + \left(-\frac{qE_0K}{m_0c^2\gamma_0}\mathbf{sin}\left(\Psi + \psi_E\right) + \frac{qE_z}{m_0c^2} \right)\frac{\partial f}{\partial \gamma} = 0\,. \tag{32}$$

Note that from now on we use $\beta_z \approx 1$. For the phase space density f we make the ansatz $f(z,\gamma,\Psi) = f_0(\gamma) + f_1(z,\gamma)\mathbf{cos}(\Psi + \psi_0)$, i.e. we assume a density modulation at the optical wavelength, growing with z (not get calculated), see Fig. 8 for illustration. The phase of this modulation is allowed to slowly depart from Ψ by ψ_0 (which is, general differently from ψ_E). In complex notation:

$$f_1(z,\gamma)\mathbf{cos}(\Psi + \psi_0) = \tfrac{f_1}{2}e^{i(\Psi+\psi_0)} + \tfrac{f_1}{2}e^{-i(\Psi+\psi_0)} = \tfrac{f_1}{2}e^{i\psi_0}e^{i\Psi} + \text{c.c.} = \tilde{f}_1(z,\gamma)e^{i\Psi} + \text{c.c.}$$

The complex amplitude $\tilde{f}_1(z,\gamma) = \tfrac{f_1}{2}e^{i\psi_0}$ of density modulation contains the slowly varying phase ψ_0. A similar ansatz is made for the space charge field $\mathbf{E}_z$:

$$\mathbf{E}_z = \mathrm{E}_z(z)\mathbf{cos}(\Psi + \psi_s) = \tilde{\mathbf{E}}_z(z)e^{i\Psi} + \text{c.c.},$$

again with its own slowly varying phase ψ_s. Vlasov's equation (32) can now be written:

$$\frac{\partial f}{\partial z} + (C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)\frac{\partial f}{\partial \Psi} + (-\frac{q\mathbf{E}_0K}{m_0c^2\gamma_0}\mathbf{sin}(\Psi + \psi_E) + \frac{q\mathbf{E}_z}{m_0c^2})\frac{\partial f}{\partial \gamma} =$$

$$= \frac{\partial \tilde{f}_1}{\partial z}e^{i\Psi} + \frac{\partial \tilde{f}_1^*}{\partial z}e^{-i\Psi} + (C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(i\tilde{f}_1e^{i\Psi} - i\tilde{f}_1^*e^{-i\Psi}) +$$

$$+\left[\frac{q\mathbf{E}_0K}{m_0c^2\gamma_0}\frac{i}{2}\left(e^{i(\Psi+\psi_E)} - e^{-i(\Psi+\psi_E)}\right) + \frac{q}{m_0c^2}(\tilde{\mathbf{E}}_z e^{i\Psi} + \tilde{\mathbf{E}}_z^* e^{-i\Psi})\right]\left(\frac{\partial f_0}{\partial \gamma} + \frac{\partial \tilde{f}_1}{\partial \gamma}e^{i\Psi} + \frac{\partial \tilde{f}_1^*}{\partial \gamma}e^{-i\Psi}\right) =$$

(using $\tilde{\mathbf{E}}_0 = \mathbf{E}_0 e^{i\psi_E}$)

$$= e^{i\Psi}\left\{\begin{aligned}&\frac{\partial \tilde{f}_1}{\partial z} + i(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)\tilde{f}_1 + (i\frac{q\tilde{\mathbf{E}}_0K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z)\frac{\partial f_0}{\partial \gamma} +\\ &\left[i\frac{q\mathbf{E}_0K}{2m_0c^2\gamma_0}(e^{i(\Psi+\psi_E)} - e^{-i(\Psi+\psi_E)}) + \frac{q}{m_0c^2}(\tilde{\mathbf{E}}_z e^{i\Psi} + \tilde{\mathbf{E}}_z^* e^{-i\Psi})\right]\frac{\partial \tilde{f}_1}{\partial \gamma}\end{aligned}\right\} + c.c. = 0$$

For this equation to hold for all phases Ψ, the expression in brackets $\{\ \}$ must vanish. Our next step in approximation assumes that the modulation amplitude does not depend on energy (see Fig. 7):

$\frac{\partial \tilde{f}_1}{\partial \gamma} = 0$. Then:

$$\boxed{\frac{\partial \tilde{f}_1(z,\gamma)}{\partial z} + \mathrm{i}\left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)\tilde{f}_1(z,\gamma) + \left(\mathrm{i}\frac{q\tilde{\mathbf{E}}_0 K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z\right)\frac{\partial f_0(\gamma)}{\partial \gamma} = 0} \quad . \tag{33}$$

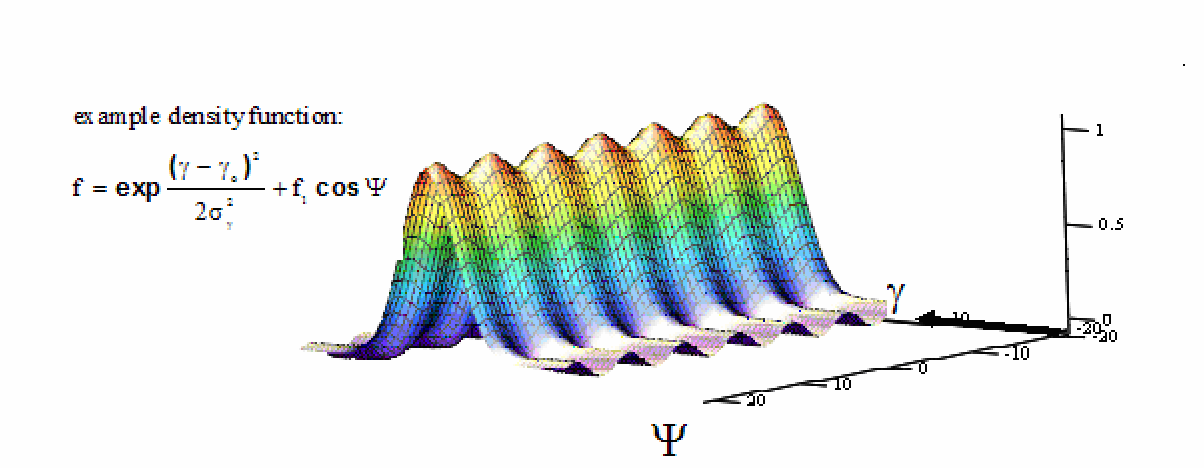

Fig. 8: Illustration of a possible phase space density function fulfilling the assumptions made here: The density modulation amplitude f_1 observed at an arbitrary location z does not depend on energy γ, and the amount of modulation in the core of the beam is small compared to the total density.

Equation (33) is a differential equation in z of the type $\frac{df(z)}{dz} + i\alpha f(z) = g(z)$, which is solved by $f(z) = \int_0^z g(z')\exp[i\alpha(z'-z)]dz'$. Thus:

$$\tilde{f}_1(z,\gamma) = \int_0^z dz' \left[i\frac{q\tilde{\mathbf{E}}_0(z')K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z') \right] \frac{\partial f_0(\gamma)}{\partial \gamma} \exp\left[i\left(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma\right)(z'-z) \right] \tag{34}$$

and $\tilde{f}_1^*(z,\gamma) = \text{c.c.}$ We can now calculate the current density:

$$j_z = \rho v_z \approx \rho c = qc\int f(z,\gamma,\Psi)d\gamma = qc\int f_0(\gamma)d\gamma + e^{i\Psi}qc\int \tilde{f}_1(z,\gamma)d\gamma + e^{-i\Psi}qc\int \tilde{f}_1^*(z,\gamma)d\gamma =$$
$$= j_0 + \tilde{j}_1 e^{i\Psi} + \tilde{j}_1^* e^{-i\Psi}, \text{with } \tilde{j}_1 = qc\int \tilde{f}_1(z,\gamma)d\gamma, \text{ etc.}$$

With these definitions, Eq. (30) reads

$$-\left[2k_L\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right] = \mu_0\frac{\mathrm{K}}{\gamma_0}\frac{\partial j_z}{\partial t}e^{i\Psi} = \mu_0\frac{\mathrm{K}}{\gamma_0}\frac{\partial\left(j_0 + \tilde{j}_1 e^{i\Psi} + \tilde{j}_1^* e^{-i\Psi}\right)}{\partial t}e^{i\Psi} .$$

We use $\Psi = (k_u + k_L)z - \omega_L t$ and assume that $\tilde{j}_1$ is 'almost' independent of time.

Then:

$$-\left[2k_L \frac{\partial}{\partial z}\tilde{\mathbf{E}}_0^*(z)\right] \approx \mu_0 \frac{\mathrm{K}}{\gamma_0}\left[\left(-\mathrm{i}\omega_L\right)\tilde{j}_1 e^{\mathrm{i}\Psi} + \left(i\omega_L\right)\tilde{j}_1^* e^{-\mathrm{i}\Psi}\right]e^{i\Psi} = \mathrm{i}\mu_0 \frac{\omega_L \mathrm{K}}{\gamma_0}\left(-\tilde{j}_1 e^{2\mathrm{i}\Psi} + \tilde{j}_1^*\right) \approx \mathrm{i}\mu_0 \frac{\omega_L \mathrm{K}}{\gamma_0}\tilde{j}_1^*$$

(neglecting the rapidly term $\tilde{j}_1 e^{2\mathrm{i}\Psi}$). Equally, $\boxed{\mathrm{i}\mu_0 \frac{\omega_L \mathrm{K}}{\gamma_0}\tilde{j}_1 = 2k_L \frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z)}$. (35)

3.3 Self-consistent description of electromagnetic field and electron distribution

We can now combine the 'field equation' (35) and the 'kinematic equation' (34) to find a self-consistent description of the evolution of the electromagnetic field and the electron density distribution:

$$\begin{aligned}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) &= \mathrm{i}\frac{\mu_0 cK}{2\gamma_0}\tilde{j}_1 = \mathrm{i}\frac{\mu_0 Kqc^2}{2\gamma_0}\int_1^\infty \tilde{f}_1(z,\gamma)d\gamma = \\ &\mathrm{i}\frac{\mu_0 Kqc^2}{2\gamma_0}\int_1^\infty d\gamma\int_0^z dz'\left[\mathrm{i}\frac{q\tilde{\mathbf{E}}_0(z')K}{2m_0c^2\gamma_0} + \frac{q}{m_0c^2}\tilde{\mathbf{E}}_z(z')\right]\frac{\partial f_0(\gamma)}{\partial\gamma}\mathbf{exp}\left[\mathrm{i}(C + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right].\end{aligned} \tag{36}$$

The problem with this equation is that it contains not only the desired complex transverse field amplitude $\tilde{\mathbf{E}}_0$, but also the longitudinal space charge field $\tilde{\mathbf{E}}_z$. Fortunately, $\tilde{\mathbf{E}}_z$ can be related to $\tilde{\mathbf{E}}_0$ in the following way: For our assumption of the space charge field $\mathbf{E}_z = \tilde{\mathbf{E}}_z(z)e^{i\Psi} + \mathrm{c.c.}$, the longitudinal component of the first Maxwell equation reads (note $\partial/\partial\mathrm{x} = \partial/\partial\mathrm{y} = 0$ in our 1D treatment):

$$\left(\nabla\times H\right)_z = 0 = j_z + \varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}_z$$

or

$$\frac{\partial}{\partial t}\mathbf{E}_z(z) = -\mu_0 c^2 j_z,$$

thus

$$\tilde{\mathbf{E}}_z(z) \approx -\frac{\mathrm{i}\mu_0 c^2}{\omega_L}\tilde{j}_1.$$

With Eq. (35) this is related to the transverse electromagnetic field:

$$\tilde{j}_1 = -\mathrm{i}\frac{2\gamma_0}{\mu_0 cK}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z),$$

thus

$$\tilde{\mathbf{E}}_z(z) \approx -\frac{2\gamma_0 c}{\omega_L K}\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z).$$

Therefore Eq. (36) becomes:

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = \mathrm{i}\frac{\mu_0 q^2 K^2 c^2}{4\gamma_0^2 m_0 c^2}\int_1^\infty d\gamma \int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{4\gamma_0^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right]\frac{\partial f_0(\gamma)}{\partial\gamma}\mathbf{exp}\left[\mathrm{i}(\mathrm{C} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right]$$

This is an integro-differential equation for the complex amplitude of the electromagnetic field. Only for few non-trivial model functions of the initial energy distribution f_0 can the solution be found analytically, using Laplace transform techniques. We restrict ourselves to the most simple case, a monoenergetic ('cold') beam: $f_0(\gamma) = n_0\delta(\gamma - \gamma_0)$, i.e. $\Delta\gamma = 0$, with charge density qn_0, i.e. $j_0 = qc\int_{-\infty}^{\infty} n_0\delta(\gamma-\gamma_0)d\gamma = qcn_0$.

Integration over energy can then be executed, using partial integration:

$$\int_1^\infty \frac{\mathrm{d}\delta(\gamma-\gamma_0)}{d\gamma}F(\gamma)d\gamma = \left[\delta(\gamma-\gamma_0)F(\gamma)\right]_1^\infty - \int_1^\infty \delta(\gamma-\gamma_0)\frac{dF(\gamma)}{d\gamma}d\gamma,$$

thus

$$\frac{\partial}{\partial z}\tilde{\mathbf{E}}_0(z) = \mathrm{i}\frac{\mu_0 n_0 q^2 K^2}{4\gamma_0^2 m_0}\times$$

$$\int_0^z dz'\int_1^\infty d\gamma\delta(\gamma-\gamma_0)\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{4\gamma^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right]\left(\mathrm{i}\frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}(z'-z)\right)\mathbf{exp}\left[\mathrm{i}(\mathrm{C} + \frac{\omega_L}{c}\frac{1+K^2}{\gamma_0^3}\Delta\gamma)(z'-z)\right] =$$

$$= -\frac{\mu_0 n_0 q^2 K^2(1+K^2)\omega_L}{4\gamma_0^5 m_0 c}\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{4\gamma_0^2 c}{\omega_L K^2}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\mathbf{exp}\left[\mathrm{iC}(z'-z)\right] =$$

$$= -\Gamma^3\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{\partial}{\partial z'}\tilde{\mathbf{E}}_0(z')\right](z'-z)\mathbf{exp}\left[\mathrm{iC}(z'-z)\right], \quad (37)$$

with abbreviations:

$\Gamma^3 = \frac{\mu_0 n_0 q^2 K^2(1+K^2)\omega_L}{4\gamma_0^5 m_0 c} = \frac{\pi j_0 K^2(1+K^2)\omega_L}{I_A c\gamma_0^5}$	Γ is called the gain parameter
$I_A = \frac{4\pi m_0 c}{\mu_0 q} = 17$ kA	is the 'Alven current'
$k_p^2 = \frac{4\pi j_0(1+K^2)}{I_A\gamma_0^3} = \Gamma^3\frac{4\gamma^2 c}{\omega_L K^2}$	k_p is the wave number of longitudinal plasma oscillation

.

Note that k_p is the only reminder of taking longitudinal space charge into account. We have ended with an ordinary integro-differential equation (37) for $\tilde{\mathbf{E}}_0$. We now derive Eq. (37) with respect to *z*:

$$\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 = -\mathrm{iC}\frac{d}{dz}\tilde{\mathbf{E}}_0 + \Gamma^3\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right]\mathbf{exp}\left[\mathrm{iC}(z'-z)\right],$$

where

$$\frac{d}{dz}\int_0^z dz' g(z)h(z') = \frac{d}{dz}\left[g(z)\int_0^z dz' h(z')\right] = \frac{d}{dz}g(z)\int_0^z dz' h(z') + g(z)h(z))$$

was used.

Finally, we derive once more and get:

$$\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 =$$

$$=\text{-i}C\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[\mathrm{i}\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] - \mathrm{i}C\Gamma^3\int_0^z dz'\left[\mathrm{i}\tilde{\mathbf{E}}_0(z') - \frac{k_p^2}{\Gamma^3}\frac{d}{dz'}\tilde{\mathbf{E}}_0(z')\right]\mathbf{exp}\left[\mathrm{i}C(z'-z)\right] =$$

$$=\text{-i}C\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[\mathrm{i}\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] - \mathrm{i}C(\mathrm{i}C\frac{d}{dz}\tilde{\mathbf{E}}_0 + \frac{d^2}{dz^2}\tilde{\mathbf{E}}_0) =$$

$$= \text{-2i}C\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \Gamma^3\left[\mathrm{i}\tilde{\mathbf{E}}_0(z) - \frac{k_p^2}{\Gamma^3}\frac{d}{dz}\tilde{\mathbf{E}}_0(z)\right] + C^2\frac{d}{dz}\tilde{\mathbf{E}}_0 \ .$$

Rearranging, we arrive at our final result: An ordinary linear third-order differential equation for the complex field amplitude $\tilde{\mathbf{E}}_0$:

$$\boxed{\frac{d^3}{dz^3}\tilde{\mathbf{E}}_0 + 2\mathrm{i}C\frac{d^2}{dz^2}\tilde{\mathbf{E}}_0 + \left(k_p^2 - C^2\right)\frac{d}{dz}\tilde{\mathbf{E}}_0 = \mathrm{i}\Gamma^3\,\tilde{\mathbf{E}}_0(z)} \ . \tag{38}$$

At the end of this derivation, Fig. 9 illustrates the major steps and approximations taking us to the final result in Eq. (38).

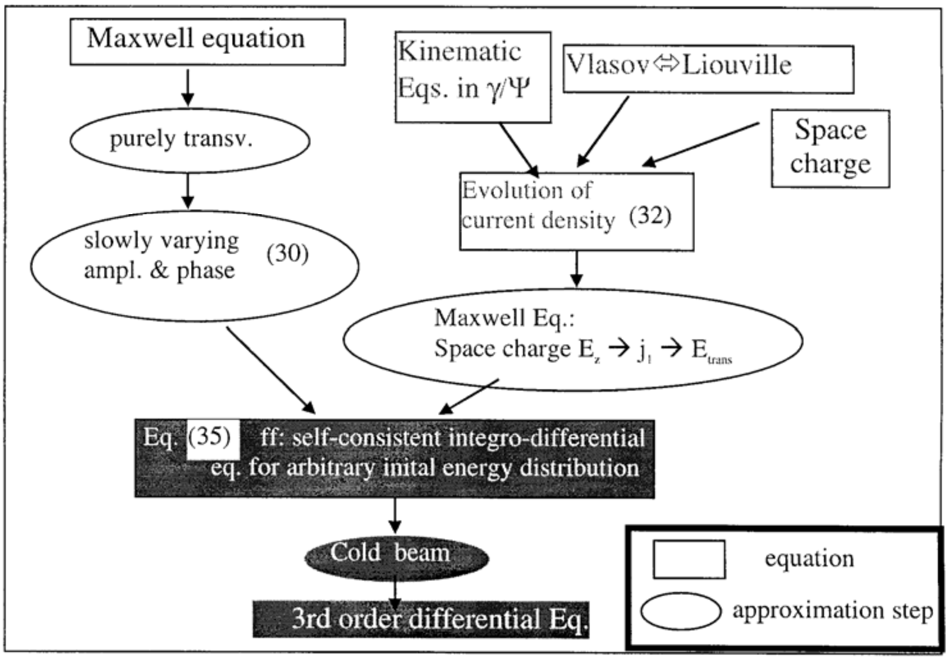

Fig. 9: Major steps to derive the 3rd order differential Eq. (38) for the high-gain free-electron laser

3.4 Solution of the high-gain FEL equation

For the solution of Eq. (38) we make the ansatz $\tilde{\mathbf{E}}_0 = A\,\mathbf{exp}(\Lambda z)$ and get the 'characteristic equation':

$$\Lambda^3 + 2\mathrm{i}C\Lambda^2 + \left(k_p^2 - C^2\right)\Lambda = \Lambda\left(\Lambda^2 + 2\mathrm{i}C\Lambda - C^2 + k_p^2\right) = \Lambda\left[\left(\Lambda + \mathrm{i}C\right)^2 + k_p^2\right] = \mathrm{i}\Gamma^3. \tag{39}$$

Equation (39) has three roots, and the general solution of Eq. (38) is constructed from three independent partial solutions:

$$\tilde{\mathbf{E}}_0(z) = A_1\,\mathbf{exp}(\Lambda_1 z) + A_2\,\mathbf{exp}(\Lambda_2 z) + A_3\,\mathbf{exp}(\Lambda_3 z). \tag{40}$$

The amplitudes A_1, A_2, A_3 are determined by the initial conditions. Since there are three free parameters, we need three independent conditions. The most practical way to specify these conditions is to specify

$$\tilde{\mathbf{E}}_0(z=0), \frac{d}{dz}\tilde{\mathbf{E}}_0(z=0), \frac{d^2}{dz^2}\tilde{\mathbf{E}}_0(z=0),$$

or, taking into account Eq. (35): $(d/dz)\tilde{\mathbf{E}}_0 \propto \tilde{j}_1$, to specify

$$\tilde{\mathbf{E}}_0(z=0), \tilde{j}_1(z=0), \frac{d}{dz}\tilde{j}_1(z=0).$$

We write Eq. (40) in the form $\tilde{\mathbf{E}}_0(z) = A_1\tilde{\mathbf{E}}_1(z) + A_2\tilde{\mathbf{E}}_2(z) + A_3\tilde{\mathbf{E}}_3(z)$, with $\tilde{\mathbf{E}}_1(z) = \mathbf{exp}(\Lambda_1 z)$, etc., and we write $(d/dz)\tilde{\mathbf{E}} = \tilde{\mathbf{E}}'$, etc. (note we will omit the index 0 to $\tilde{\mathbf{E}}_0$ in the following). The general solution, including its first and second derivatives, can then be written in a matrix form:

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \mathrm{A}_1 \\ \mathrm{A}_2 \\ \mathrm{A}_3 \end{pmatrix},$$

where the index z means that the matrix elements are taken at longitudinal position z. Since Λ_1, Λ_2, Λ_3 are known from the characteristic equation (39), all matrix elements are known. Writing the initial condition in the form

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0},$$

we can calculate A_1, A_2, A_3 from

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_{z=0}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} .$$

Thus,

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_{z=0}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0}$$

or, using $\tilde{\mathbf{E}}_1(z) = \mathbf{exp}(\Lambda_1 z)$, $\tilde{\mathbf{E}}_1'(z) = \Lambda_1 \mathbf{exp}(\Lambda_1 z)$, etc.,

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} 1 & 1 & 1 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} . \quad (41)$$

Using the explicit expression for the inverse matrix, Eq. (41) reads

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{\Lambda_2\Lambda_3}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} & -\frac{\Lambda_2+\Lambda_3}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} & \frac{1}{(\Lambda_1-\Lambda_2)(\Lambda_1-\Lambda_3)} \\ \frac{\Lambda_1\Lambda_3}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} & -\frac{\Lambda_1+\Lambda_3}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} & \frac{1}{(\Lambda_2-\Lambda_1)(\Lambda_2-\Lambda_3)} \\ \frac{\Lambda_2\Lambda_1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} & -\frac{\Lambda_2+\Lambda_1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} & \frac{1}{(\Lambda_3-\Lambda_2)(\Lambda_3-\Lambda_1)} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} . \quad (42)$$

3.5 Solution for the case $C = k_p = 0$

To be more specific, we now investigate the most simple case. No detuning, i.e. all the electrons have the same energy, and this energy meets exactly the resonance condition: $C = 0$. Also, we assume negligible impact of space charge, i.e.

$$k_p^2 = \frac{4\pi j_0 (1+K^2)}{I_A \gamma_0^3} \to 0 .$$

The validity of this latter condition is a little more difficult to verify and should be considered with care for each individual case. This condition tends to be valid at very high beam energy γ_0. With these assumptions, the three roots of Eq. (39) are:

$$\Lambda^3 = \mathrm{i}\Gamma^3 \Rightarrow \Lambda_1 = -\mathrm{i}\Gamma; \quad \Lambda_2 = \frac{\mathrm{i}+\sqrt{3}}{2}\Gamma; \quad \Lambda_3 = \frac{\mathrm{i}-\sqrt{3}}{2}\Gamma . \quad (43)$$

The general solution is thus:

$$\tilde{\mathbf{E}}(z) = A_1\tilde{\mathbf{E}}_1(z) + A_2\tilde{\mathbf{E}}_2(z) + A_3\tilde{\mathbf{E}}_3(z) = A_1\,\mathbf{exp}(\Lambda_1 z) + A_2\,\mathbf{exp}(\Lambda_2 z) + A_3\,\mathbf{exp}(\Lambda_3 z)$$
$$= A_1\,\mathbf{exp}(-\mathrm{i}\Gamma z) + A_2\,\mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right) + A_3\,\mathbf{exp}\left(\frac{\mathrm{i}-\sqrt{3}}{2}\Gamma z\right).$$

Obviously, all contributions to this solution either vanish with increasing z, or they oscillate, except for the one containing $A_2\,\mathbf{exp}\left(\frac{\sqrt{3}}{2}\Gamma z\right)$. For an undulator much longer than $1/\Gamma$, this part of the solution will dominate.

Using $\Lambda_1, \Lambda_2, \Lambda_3$ from Eq. (43), Eq. (42) reads now (note $1+\mathrm{i}\sqrt{3} = 2\,\mathbf{exp}\,\mathrm{i}\frac{\pi}{3}$):

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{1}{3} & \frac{\mathrm{i}}{3\Gamma} & -\frac{1}{3\Gamma^2} \\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-\mathrm{i}\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-\mathrm{i}\frac{\pi}{3}\right) \\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(\mathrm{i}\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(\mathrm{i}\frac{\pi}{3}\right) \end{pmatrix} \cdot \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0}, \tag{44}$$

which we will evaluate in the following for two different initial conditions.

3.5.1 Seeding by external electromagnetic wave at the undulator entrance

First, we consider the case of an external ('seeding') electromagnetic wave (with amplitude $\mathbf{E}_{ext}$) existing at the undulator entrance, but no initial longitudinal modulation of the electron beam, i.e. $\tilde{j}_1(z=0) = 0$. Consequently,

$$\tilde{\mathbf{E}}(z=0) = E_{ext}, \tilde{j}_1(z=0) = 0,\ \frac{d}{dz}\tilde{j}_1(z=0) = 0 \ \rightarrow \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} = \begin{pmatrix} \mathbf{E}_{ext} \\ 0 \\ 0 \end{pmatrix}.$$

Thus:

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2} \\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right) \\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_{ext} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' & \tilde{\mathbf{E}}_2' & \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' & \tilde{\mathbf{E}}_2'' & \tilde{\mathbf{E}}_3'' \end{pmatrix}_z \cdot \begin{pmatrix} \frac{1}{3}\mathbf{E}_{ext} \\ \frac{1}{3}\mathbf{E}_{ext} \\ \frac{1}{3}\mathbf{E}_{ext} \end{pmatrix}$$
$$= \frac{1}{3}\mathbf{E}_{ext}\begin{pmatrix} \tilde{\mathbf{E}}_1 + \tilde{\mathbf{E}}_2 + \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}_1' + \tilde{\mathbf{E}}_2' + \tilde{\mathbf{E}}_3' \\ \tilde{\mathbf{E}}_1'' + \tilde{\mathbf{E}}_2'' + \tilde{\mathbf{E}}_3'' \end{pmatrix} = \frac{1}{3}\mathbf{E}_{ext}\begin{pmatrix} \mathbf{exp}(\Lambda_1 z) + \mathbf{exp}(\Lambda_2 z) + \mathbf{exp}(\Lambda_3 z) \\ \Lambda_1\,\mathbf{exp}(\Lambda_1 z) + \Lambda_2\,\mathbf{exp}(\Lambda_2 z) + \Lambda_3\,\mathbf{exp}(\Lambda_3 z) \\ \Lambda_1^2\,\mathbf{exp}(\Lambda_1 z) + \Lambda_2^2\,\mathbf{exp}(\Lambda_2 z) + \Lambda_3^2\,\mathbf{exp}(\Lambda_3 z) \end{pmatrix}$$

Explicitly, the solution for $\tilde{\mathbf{E}}(z)$ is

$$\tilde{\mathbf{E}}(z) = \frac{1}{3}\mathbf{E}_{ext}\left[\mathbf{exp}(-\mathrm{i}\Gamma z) + \mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right) + \mathbf{exp}\left(\frac{\mathrm{i}-\sqrt{3}}{2}\Gamma z\right)\right].$$

As mentioned before, for $z \gg 1/\Gamma$, the solution with Λ_2 dominates:

$$\boxed{\tilde{\mathbf{E}}(z) = \frac{1}{3}\mathbf{E}_{ext}\,\mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right)}\,. \tag{45}$$

The power gain, defined by $G = \frac{|\tilde{\mathbf{E}}|^2}{\mathbf{E}_{ext}^2}$, is calculated from Eq. (45) and results in:

$$G = \frac{|\tilde{\mathbf{E}}|^2}{\mathbf{E}_{ext}^2} = \frac{1}{9}\left[1 + 4\,\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z\left(\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z + \mathbf{cos}\frac{3}{2}\Gamma z\right)\right]. \tag{46}$$

For $z \gg 1/\Gamma$, this simplifies to

$$\boxed{G = \frac{1}{9}\,\mathbf{exp}\sqrt{3}\Gamma z}\,. \tag{47}$$

The factor 1/9 describes the efficiency at which the incoming (‘seeding’) electromagnetic field couples to the FEL gain process. Figure 10 shows a plot of Eqs. (46, 47) as a function of Γz, indicating that, indeed, the gain grows exponentially according to Eq. (47) for $z \gg 1/\Gamma$. The e-folding length of radiation power is called (power) gain length L_G:

$$L_G = \frac{1}{\sqrt{3}\Gamma} = \frac{1}{\sqrt{3}}\left(\frac{I_A c\gamma^5}{\pi j_0 K^2(1+K^2)\omega_L}\right)^{1/3}.$$

Using $\omega_L = \frac{4\pi c\gamma^2}{\lambda_u(1+K^2)}$ and expressing the current density $j_0 \approx \frac{\hat{I}}{\pi\sigma_r^2}$ in terms of peak current $\hat{I}$ and beam cross section $\pi\sigma_r^2$, this can be written

$$\boxed{L_G = \frac{1}{\sqrt{3}}\left(\frac{I_A\gamma^3\sigma_r^2\lambda_u}{4\pi\hat{I}K^2}\right)^{1/3}}\,. \tag{48}$$

Note that some authors use the e-folding length for the field amplitude which is $2L_G$. Another parameter widely used is the dimensionless ‘FEL-parameter’ ρ:

$$\rho = \frac{\lambda_u \Gamma}{4\pi} = \frac{1}{4\pi\sqrt{3}} \frac{\lambda_u}{L_G} = \frac{1}{4\pi\sqrt{3}} \frac{1}{N_{Gain}} . \tag{49}$$

Within one power gain length N_{Gain} is the number of undulator periods.

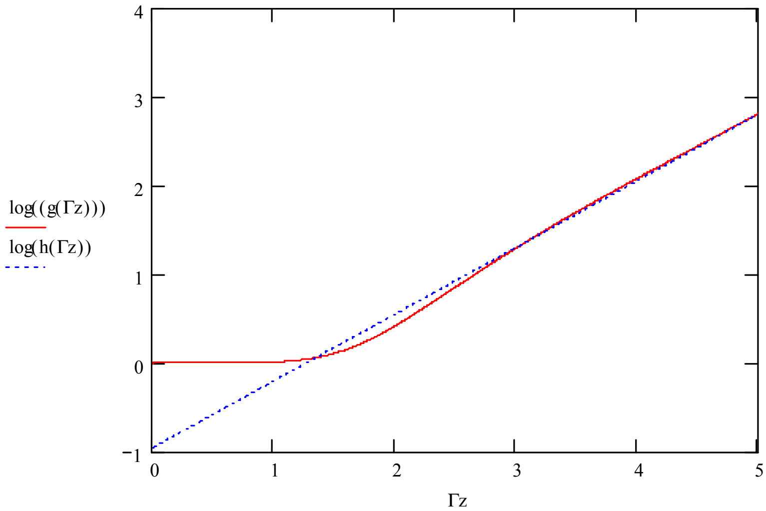

Fig. 10: Plot of the power gain of a high-gain FEL, starting with a seeding electromagnetic wave, see Eq. (46). The dotted line is the asymptotic solution Eq. (47) for $z \gg 1/\Gamma$. The vertical scale is logarithmic.

3.5.2 Initial longitudinal modulation of electron-beam density

As a second example, consider when instead of an external electromagnetic wave at the undulator entrance there is a longitudinal current modulation of the electron beam at the radiation wavelength, which is assumed to be stationary at the beginning:

$$\tilde{\mathbf{E}}(z=0)=0,\ \ \tilde{j}_1(z=0) \neq 0\ ,\ \frac{d}{dz}\tilde{j}_1(z=0)=0 .$$

Thus:

$$\tilde{\mathbf{E}}'(z=0) = i\mu_0 \frac{cK}{2\gamma_0} \tilde{j}_1(z=0),\ \tilde{\mathbf{E}}''(z=0)=0 \ \text{ and } \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_{z=0} = \begin{pmatrix} 0 \\ i\mu_0 \frac{cK}{2\gamma_0} \tilde{j}_1 \\ 0 \end{pmatrix}_{z=0} = \begin{pmatrix} 0 \\ \tilde{\mathbf{E}}'_0 \\ 0 \end{pmatrix} . \tag{50}$$

Therefore:

$$\begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{E}}' \\ \tilde{\mathbf{E}}'' \end{pmatrix}_z = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}'_1 & \tilde{\mathbf{E}}'_2 & \tilde{\mathbf{E}}'_3 \\ \tilde{\mathbf{E}}''_1 & \tilde{\mathbf{E}}''_2 & \tilde{\mathbf{E}}''_3 \end{pmatrix}_z \begin{pmatrix} \frac{1}{3} & \frac{i}{3\Gamma} & -\frac{1}{3\Gamma^2} \\ \frac{1}{3} & \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(-i\frac{\pi}{3}\right) \\ \frac{1}{3} & \frac{-1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right) & \frac{1}{3\Gamma^2}\mathbf{exp}\left(i\frac{\pi}{3}\right) \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{\mathbf{E}}'_0 \\ 0 \end{pmatrix}_{z=0} = \begin{pmatrix} \tilde{\mathbf{E}}_1 & \tilde{\mathbf{E}}_2 & \tilde{\mathbf{E}}_3 \\ \tilde{\mathbf{E}}'_1 & \tilde{\mathbf{E}}'_2 & \tilde{\mathbf{E}}'_3 \\ \tilde{\mathbf{E}}''_1 & \tilde{\mathbf{E}}''_2 & \tilde{\mathbf{E}}''_3 \end{pmatrix}_z \begin{pmatrix} \frac{i}{3\Gamma}\tilde{\mathbf{E}}'_0 \\ \frac{1}{3\Gamma}\mathbf{exp}\left(-i\frac{\pi}{6}\right)\tilde{\mathbf{E}}'_0 \\ -\frac{1}{3\Gamma}\mathbf{exp}\left(i\frac{\pi}{6}\right)\tilde{\mathbf{E}}'_0 \end{pmatrix} .$$

Explicitly, the solution for $\tilde{\mathbf{E}}(z)$ is

$$\tilde{\mathbf{E}}(z)=\frac{1}{3\Gamma}\tilde{\mathbf{E}}'_0\left[\mathrm{i}\,\mathbf{exp}(\Lambda_1 z)+\mathbf{exp}\left(-\mathrm{i}\frac{\pi}{6}\right)\mathbf{exp}(\Lambda_2 z)-\mathbf{exp}\left(\mathrm{i}\frac{\pi}{6}\right)\mathbf{exp}(\Lambda_3 z)\right]. \tag{51}$$

Again, for $z \gg 1/\Gamma$, the solution with Λ_2 dominates:

$$\tilde{\mathbf{E}}(z \gg \frac{1}{\Gamma}) \propto \mathbf{exp}\left(\frac{\mathrm{i}+\sqrt{3}}{2}\Gamma z\right),$$

i.e. we get an exponential growth with the same e-folding length as in the seeding case. The main point is that we do not need any input seeding electromagnetic wave: a current modulation at the optical, resonant wavelength is as good for starting the process, no matter how small the current modulation.

From Eq. (50), the radiation power as a function of z is calculated:

$$P(z) \propto \left|\tilde{\mathbf{E}}(z)\right|^2 \propto \mathbf{cos}\frac{3}{2}\Gamma z\cdot\mathbf{cosh}\frac{\sqrt{3}}{2}\Gamma z-\sqrt{3}\,\mathbf{sin}\frac{3}{2}\Gamma z\cdot\mathbf{sinh}\frac{\sqrt{3}}{2}\Gamma z+\mathbf{cosh}\sqrt{3}\Gamma z. \tag{52}$$

The asymptotic behaviour for $z \gg 1/\Gamma$ is $P(z) \propto \mathbf{exp}\sqrt{3}\Gamma z$, very much like in the seeding case. Figure 11 illustrates both Eq. (52) and its asymptotic behaviour.

It is interesting to note that, like in the seeding case, the exponential growth of the electromagnetic field starts only after approximately three gain lengths, a distance often called ‘lethargy regime’.

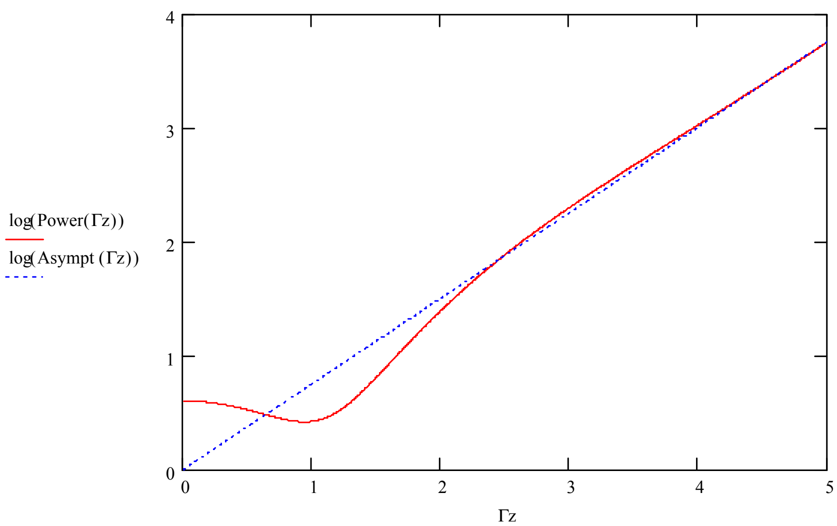

Fig. 11: Plot of the power gain of a high-gain FEL, starting with a longitudinal current modulation of the electron beam at the radiation wavelength, see Eq. (50). The dotted line is the asymptotic solution for $z \gg 1/\Gamma$. The vertical scale is logarithmic.

3.6 Resonance width

In the previous section we assumed the electrons all had the same energy, which meets exactly the resonance condition $C = 0$. Analysis of the characteristic equation (39) for $k_p = 0$ and $C \neq 0$ is straightforward algebra.

1. Maximum gain occurs for ON-resonance operation (i.e. for $C = 0$). It is important to point out that this behaviour is fundamentally in contrast to the low-gain case, where initially no gain was found for particles on resonance energy, see Fig. 4.

2. The gain drops significantly when $|C|$ is increased to values corresponding to $\frac{\Delta\gamma}{\gamma} = \rho$.

Because of $\lambda_L \propto \frac{1}{\gamma^2}$, this means the bandwidth of a high-gain FEL is

$$\boxed{\frac{\Delta\lambda_L}{\lambda_L} = 2\frac{\Delta\gamma}{\gamma} = 2\rho}. \tag{53}$$

Particles outside this energy window do not contribute to the gain process constructively. Therefore, the relative energy spread with the electron bunch should be smaller than ρ:

$$\frac{\Delta\gamma}{\gamma} \leq \rho .$$

This requirement is a serious technical challenge for FELs operating at low ρ-values. This is the case for very short wavelength λ_L in tendency. For instance, for the LCLS X-ray FEL presently under construction at SLAC, ρ is approximately 10^{-4}. A comparison of the theoretically expected bandwidth with measurements taken at the short-wavelength FEL at DESY is illustrated in Fig. 10.

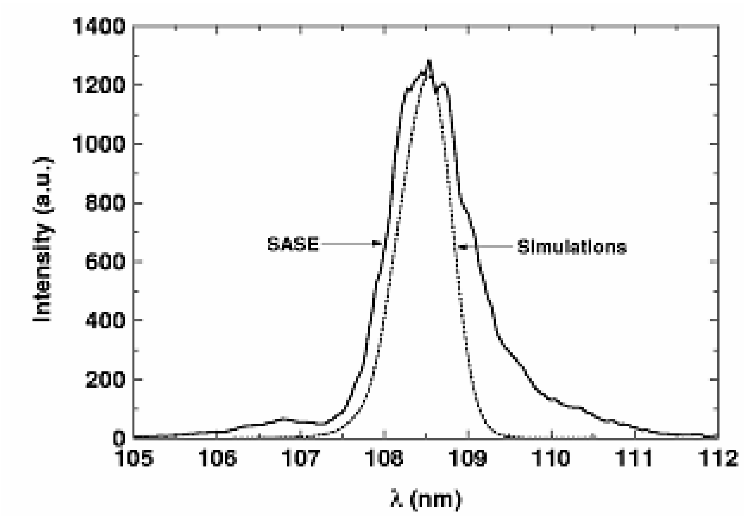

Fig. 12: Wavelength spectrum of the central radiation cone measured at the high-gain FEL at DESY [7], called TTF FEL. The dotted line represents the theoretically expected line width.

The same facts can be formulated differently. A high-gain FEL acts as a narrow-band amplifier with bandwidth $\frac{\Delta\omega}{\omega} \leq 2\rho$.

3.7 Laser saturation

The exponential growth of radiation power will not proceed forever. It comes to an end when the electron beam current is perfectly modulated at the optical wavelength. The precise behaviour of the high-gain FEL in this saturation regime cannot be treated within our analysis because our linear approximation is based on the assumption $\frac{|\tilde{j}_1|}{j_0} \ll 1$. Some typical features of the saturation regime follow. The electrons lose so much energy that they fall out of the resonance condition. Bunching and motion in phase space may even cause the electromagnetic field to pump back some energy to the electron beam. A potential cure for this is undulator tapering, i.e. increasing the *K* parameter to compensate for the loss of electron energy. Also, the energy spread of the electron beam increases (thus the frequency spread of radiation). In any case, the analysis of the non-linear saturation behaviour needs numerical simulation and is beyond the scope of this paper.

However, we are able to perform a simple estimate of the radiation power at saturation: Let us assume $|\tilde{j}_1| = j_0$, i.e. full modulation. With Eq. (35) we estimate the field amplitude at saturation with the assumption that the major part of radiation is generated within the last gain length:

$$\left|\tilde{\mathbf{E}}_0\left(z = L_G\right)\right| \approx \frac{d}{dz}\tilde{\mathbf{E}}_0\left(L_G\right) \times L_G \approx \mu_0 \frac{cK}{2\gamma_0} j_0 L_G .$$

Plugging L_G in from Eq. (48) yields

$$\boxed{P_{sat} = \frac{\varepsilon_0 c}{2}\left|\mathbf{E}\right|^2 \times Area \approx \frac{\varepsilon_0 c \sigma_r^2}{2}\left|\mathbf{E}_{sat}\right|^2 \approx \frac{\mu_0 c}{120}\left(\frac{\hat{I}^2 I_A K \lambda_u}{\sigma_r}\right)^{2/3}} . \tag{54}$$

It is interesting to note that, within this approximation, the saturation power depends neither on beam energy γ nor on radiation wavelength λ_L, very much in contrast with the power of spontaneous undulator radiation. Typical numbers may be:

$$\hat{I} = 1000\ \text{A}, \quad \text{K} = 1, \quad \lambda_u = 0.03\ \text{m}, \quad \sigma_r = 0.1\ \text{mm} \rightarrow P_{sat} \approx 2\ \text{GW} .$$

Figure 13 illustrates the onset of FEL saturation at a power level of 1 GW observed at the TTF FEL at DESY, Hamburg, with parameters close to these values.

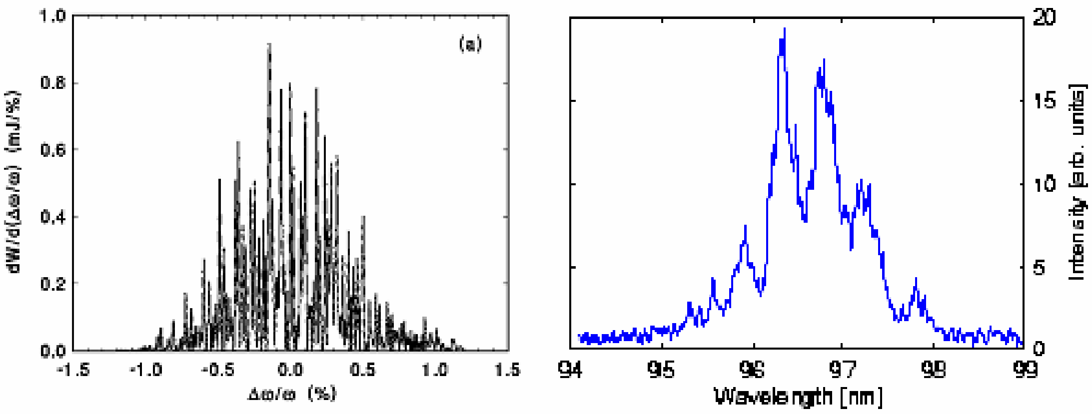

Fig. 13: Since SASE FELs start from shot noise, the output radiation spectrum is expected to be noisy. In the extreme case of the numerical simulation shown on the left-hand side, there is a very large number of spikes (large number of 'longitudinal modes') which fluctuate from electron bunch to electron bunch in intensity within the bandwidth of the FEL. The plot on the right-hand side shows measurement at TTF FEL of a single shot spectrum with mode number $M \approx 6$. The envelope of this spectrum corresponds to about $\Delta\lambda_{ph}/\lambda_{ph} \approx 0.015$, in agreement with Fig. 12.

The amount of electron beam power converted to FEL output radiation is called power efficiency and is given by:

$$\boxed{\frac{P_{sat}}{P_{beam}} = \frac{P_{sat}}{\gamma_o m_0 c^2 \hat{I}/q} \approx \rho},$$

i.e. it is just given by the FEL parameter ρ. As a rule of thumb, saturation occurs after 20 power gain lengths. For the most challenging high-gain FEL projects aiming at sub-nanometre wavelengths (e.g. LCLS/SLAC, and the European XFEL/DESY), L_{sat} will be as long as 100–200 m.

3.8 Start-up from noise: Self-Amplified Spontaneous Emission (SASE)

Section 3.5.2 demonstrates that a small arbitrary current modulation of the electron beam current at the entrance of the undulator will suffice to start the exponential FEL process. Of course, this modulation must be at the resonant radiation wavelength λ_L, determined by the electron energy and undulator parameters λ_u, K, see Eq. (12). For very short wavelengths (micrometres or nanometres), this is very difficult to achieve. In fact, because of the narrow-bandpass property described in the previous section, it would be sufficient if the longitudinal electron bunch profile contained Fourier components at λ_L. However, for normal electron bunch lengths of some 1 mm and λ_L well below a micrometre, this is (practically) not the case.

Making use of the fact that the electron beam is actually made up of many point-like charges (i.e. electrons) randomly distributed in space and time provides a very elegant way around this[4]. Such a random distribution generates a white noise spectrum of current modulation, which always contains

[4] There is simple proof that this random distribution really exists: It is the basis for spontaneous undulator radiation. As long as the observed characteristics of spontaneous undulator radiation agree with theoretical expectations, we can safely assume that electrons are distributed randomly.

some spectral contribution within the FEL bandwidth. This principle was first proposed by Kondratenko and Saldin in 1980 [8] and is widely known as the 'Self-Amplified Spontaneous Emission' mode (SASE) of high-gain FELs. It is most attractive for very short wavelengths, where no mirrors are available to construct an optical cavity, and no external lasers are available to produce a sufficiently powerful input wave. Tuning FEL output wavelength is extremely simple in the SASE mode: Just change the electron energy (or, if you prefer, the undulator *K* parameter) accordingly, and the SASE process 'automatically' selects the correct modulation wavelength from shot noise.

A characteristic property of SASE FELs is a noisy output radiation spectrum, because the FEL amplifies a part of the shot noise spectrum. Figure 13 illustrates an extreme case of a noisy output spectrum. The frequency width $\Delta\omega$ of the individual spikes in the output spectrum is determined by the length of the electron bunch l_{bunch} according to $\Delta\omega = 2\pi c/l_{bunch}$, i.e. the Fourier transform limit given by the bunch length.

Another important quantity is the number $n_c = L_G/\lambda_u$ of undulator periods within one gain length. Since the radiation pulse slips by one wavelength per undulator period with respect to the electron bunch, it is this quantity n_c that determines the number of wavelengths where coherence is expected within the FEL process. The quantity $l_{coh} \approx n_c \cdot \lambda_L$ is called coherence length. Using Eq. (49), it can be written $l_{coh} \approx \lambda_L/\pi\rho$ (note the factor π comes from a more detailed analysis, Ref. [2]). We would expect that the quantity $\lambda_L/l_{coh} \approx \pi\rho$ determine the relative bandwidth of the FEL, which is indeed the case, see Eq. (53). If this quantity is larger than $\Delta\omega$, it determines the envelope spectrum containing *M* spikes in statistical average. In terms of l_{coh}, it is the number of the coherence lengths l_{coh} within the bunch length that determines the average number $M = l_{bunch}/l_{coh}$ of spikes within the FEL output spectrum. *M* is called the number of longitudinal modes.

How can we calculate the initial conditions the FEL process is subjected to by the shot noise? One way is to estimate the effective current modulation within the bandwidth $\Delta\lambda_{ph}/\lambda_{ph} = 2\rho$ and use this value as 'initial longitudinal current modulation' in the analysis described in Section 3.5.2. Another way is to calculate the equivalent input power generated within the first gain length by shot noise and use this value as an external 'seed wave' in Section 3.5.1. This 'effective input power' P_{sh} of shot noise can be estimated at

$$P_{sh} \approx \frac{3}{N_c\sqrt{\pi \ln N_c}}\rho P_{beam} \tag{55}$$

(see Ref. [2], Eq. (6.95)). Here, P_{beam} is the electron beam power and N_c is 0.5 times the number of electrons within the coherence length. The power gain of a SASE FEL at saturation can be estimated from Eqs. (54) and (55) as:

$$G_{sat} = \frac{P_{sat}}{P_{sh}} = \frac{\rho P_{beam}}{P_{sh}} \approx \frac{1}{3}N_c\sqrt{\pi \ln N_c}\ ,$$

i.e. it is roughly given by the number of electrons in the cooperation length. The quantity P_{sh} is relevant in two ways:

1. An available estimate for P_{sh} allow us to compare the theoretical SASE model with measurements. Figure 14 shows the exponential gain observed at the SASE FEL at DESY. Within the first five gain lengths, the measured radiation power is dominated by the spontaneous undulator radiation, so that the start-up process and the lethargy regime cannot be observed directly. However, if the exponential gain curve is (exponentially) extrapolated down to the beginning of the undulator, it hits the vertical axis at a value very much in

agreement with Eq. (55). Since P_{sh} is the power amplified by the high-gain FEL, Fig. 14 indicates a total power gain by 8 orders of magnitude, about the number of electrons in the cooperation length. This does not mean that the FEL output power exceeds the power of spontaneous undulator radiation by this factor. In contrast, the power of spontaneous undulator radiation may even be comparable to FEL saturation power, but is radiated into a much wider spectrum and opening angle.

2. If one plans to improve the spectral purity of the FEL by using a seeding laser, Eq. (55) provides a lower limit of its required power. If the seed laser power does not exceed P_{sh}, the output radiation will still be determined by shot noise rather than by the seed laser spectrum.

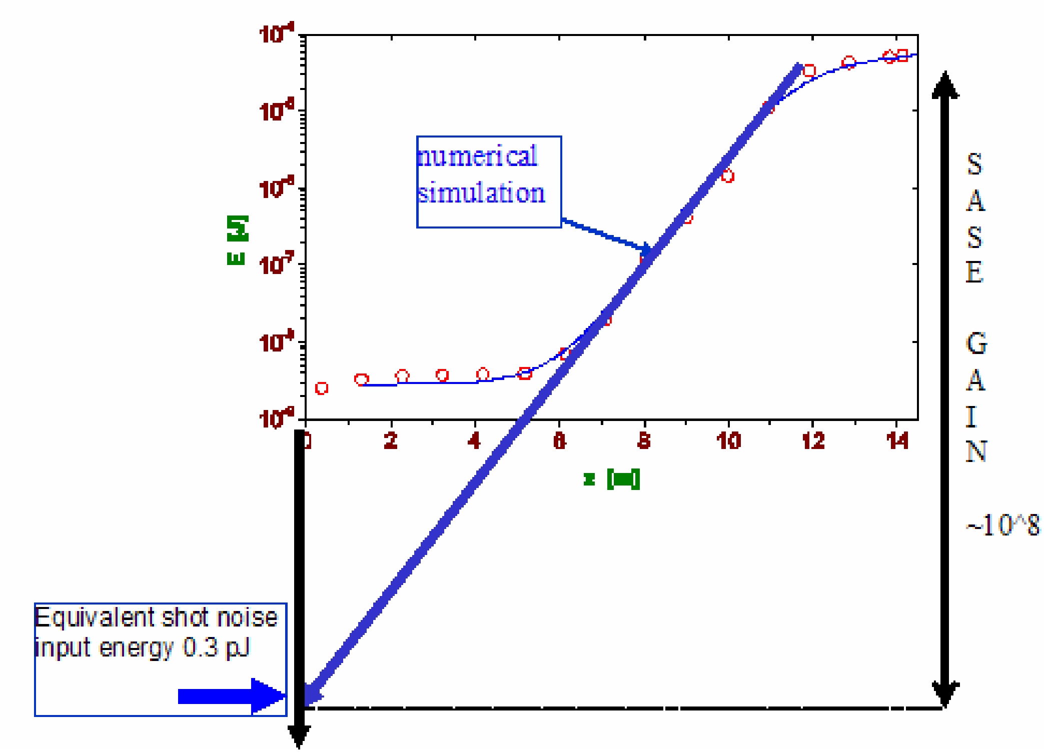

Fig. 14: Energy in the radiation pulse as a function of longitudinal position in the undulator measured at the SASE FEL at DESY at λ_L = 98 nm (dots). The vertical scale is logarithmic. The solid line is the theoretical expectation. If the exponential gain curve is (exponentially) extrapolated down (blue arrow) to the beginning of the undulator, it hits the vertical axis at a value very much in agreement with Eq. (55).

3.9 3D effects

Analysis of effects due to the finite transverse size of both the radiation and the electron beam goes beyond the scope of this article. However, sone 3D effects have a tremendous practical relevance and will be summarized here in a semi-quantitative way.

3.9.1 Transverse overlap between electron beam and electromagnetic radiation

The most prominent 3D issue is that the FEL gain process requires complete transverse overlap between the electron beam and the radiation beam during the complete passage of the undulator to ensure that the interaction between electromagnetic wave and the electron beam takes place as described. Taking into account that the transverse r.m.s. beam size is 100 µm or less (see below) for a short-wavelength FEL, the electron orbit must not depart from a perfectly straight line by more than some 10 μm over several gain lengths. This puts stringent tolerances on undulator field errors and is technically difficult both to realize and to verify.

3.9.2 Diffraction

Because of diffraction, even a perfectly coherent plane wave grows in transverse size after a while if it is collimated to a transverse radius of σ_r. The distance $L_R = \pi\sigma_r^2/\lambda_L$ after which the radiation beam is grown by approximately a factor of 2 is called the Rayleigh length and provides an estimate of the opening angle $\sigma_{\sphericalangle}$ of the radiation (Fig. 15):

$$\sigma_{\sphericalangle} \approx \frac{2\sigma_r}{L_R} \approx \frac{\lambda_L}{2\sigma_r}.$$

An equivalent estimate comes from the transverse phase space volume covered by a perfectly coherent source known to be

$$\sigma_r \sigma_{\sphericalangle} = \frac{\lambda_L}{2},$$

thus

$$\sigma_{\sphericalangle} = \frac{\lambda_L}{2\sigma_r}.$$

Typical numbers for the LCLS project are $\lambda_L \approx 10^{-10}$ m, $\sigma_r \approx 30$ μm, yielding $\sigma_{\sphericalangle} \approx 2$ μrad. It is interesting to note that this value is much smaller than the characteristic opening angle of undulator radiation $1/\gamma \approx 30$ μrad. The reason for this is that FEL radiation is not single-charge radiation but is a product of coherent superposition of radiation coming from many electrons distributed in the longitudinal direction, much like an array of antennas is able to generate a directional characteristic of radio wave emission.

Within our FEL analysis we have implicitly assumed that the electromagnetic wave is transversely coherent during the entire process. This is certainly not the case for SASE. The SASE FEL starts with many transverse optical modes. Since the axial mode achieves the highest gain, it reaches saturation first, so that 'normally' at saturation the radiation is almost fully coherent.

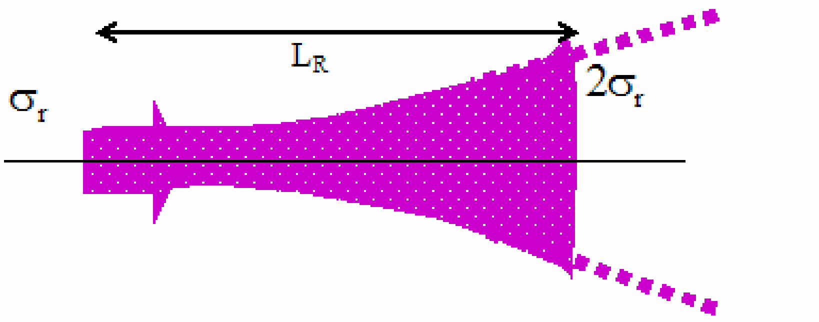

Fig. 15: Sketch of the growth of the transverse size of the radiation beam due to diffraction within a distance called the Rayleigh length L_R.

3.9.3 *Emittance of the electron beam*

The emittance of the electron beam introduces a longitudinal velocity spread in the electron beam very much as energy spread does. Thus, in terms of FEL gain, electron emittance is equivalent to additional energy spread. The equivalent energy spread is

$$\left.\frac{\Delta\gamma}{\gamma}\right|_{eff} \approx \frac{\gamma^2\varepsilon}{\beta\left(1+K^2\right)}$$

(β is the Twiss parameter of electron focusing). With the condition $\Delta\gamma/\gamma < \rho$ derived from Eq. (53) this gives a limit for the beam emittance:

$$\varepsilon \approx \frac{\beta\left(1+K^2\right)}{\gamma^2}\left.\frac{\Delta\gamma}{\gamma}\right|_{\grave{e}ff} < \frac{\beta\left(1+K^2\right)}{2\gamma^2}\rho\,, \tag{56}$$

where the factor 2 makes sure the emittance contributes less than 50% of the total energy spread budget.

A second condition comes from the diffraction effect: We want to maintain both complete overlap of electron beam and radiation (calling for long L_R thus large σ_r) AND maximum possible gain (calling for small σ_r, thus small L_R). The best compromise is $L_R \approx L_g$, thus

$$L_R = \frac{\pi\sigma_r^2}{\lambda_L} = \frac{\pi\beta\varepsilon}{\lambda_L} \approx L_G = \frac{1}{4\pi\sqrt{3}}\frac{\lambda_u}{\rho}\,.$$

With the help of Eq. (46), ρ can be eliminated, yielding

$$\varepsilon < \frac{\lambda_{ph}}{\sqrt{4\sqrt{3}\pi}} \approx \frac{\lambda_{ph}}{4\pi}\,.$$

$\boxed{\varepsilon < \frac{\lambda_{ph}}{4\pi}}$ is a rather challenging condition for λ_L in the nanometer range.

3.10 Velocities

When we introduced the complex field amplitude $\tilde{\mathbf{E}}_0(z) = \mathbf{E}_0(z)\exp i\psi_E$ in Section 3.1, we intentionally introduced an additional phase ψ_E. This slowly varying phase describes the slippage of the electromagnetic phase with respect to a free wave propagating at phase velocity *c*. We can determine $\cos\psi_E$ by (see. Fig. 16):

$$\cos\psi_E = \frac{\Re\tilde{\mathbf{E}}}{\left|\tilde{\mathbf{E}}\right|} = \frac{\cos(\Gamma z) + 2\cos\left(\frac{\Gamma z}{2}\right)\cdot\cosh\left(\frac{\sqrt{3}}{2}\Gamma z\right)}{\sqrt{1+4\cosh\frac{\sqrt{3}}{2}\Gamma z\left(\cosh\frac{\sqrt{3}}{2}\Gamma z + \cos\frac{3}{2}\Gamma z\right)}}\,.$$

For $z \gg 1/\Gamma$ this reads

$$\cos\psi_E = \frac{\Re\tilde{\mathbf{E}}}{|\tilde{\mathbf{E}}|} = \frac{\Re\exp\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)}{\left|\exp\left(\frac{i+\sqrt{3}}{2}\Gamma z\right)\right|} = \Re\exp\left(\frac{i}{2}\Gamma z\right) = \cos\frac{\Gamma z}{2},$$

i.e. $\psi_E = \dfrac{\Gamma z}{2}$ for $z \gg 1/\Gamma$.

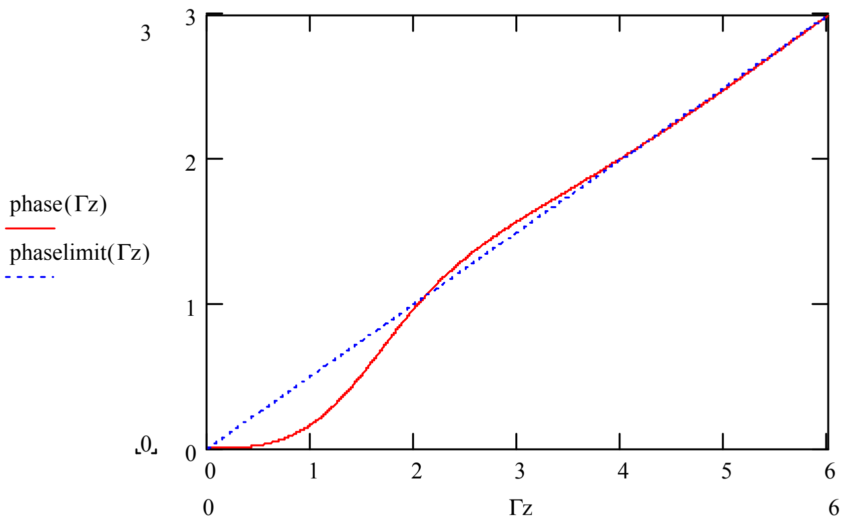

Fig. 16: Development of the slowly varying phase ψ_E as a function of the longitudinal coordinate z, normalized tio the gain parameter Γ. ψ_E describes the slippage of the electromagnetic phase with respect to a free wave propagating at phase velocity c.

We can now calculate the phase velocity of the electromagnetic wave during the FEL process:

$$v_{ph} = \frac{\omega}{k} = \frac{\omega}{k_L + \dfrac{\Gamma}{2}},$$

i.e. is reduced by

$$c - v_{ph} \approx c - \frac{\omega}{k_L}\left(1 - \frac{\Gamma}{2k_L}\right) = \frac{\Gamma}{2k_L}c$$

with respect to a free electromagnetic wave.

Similarly, we can calculate the phase velocity of density modulation. With Eq. (35), and using Eq. (44):

$$j_z = j_0 + \tilde{j}_1 e^{i\Psi} + \text{c.c.} = j_0 - i\frac{2\gamma_0}{\mu_0 cK}\frac{1}{3}\mathbf{E}_{ext}\,\Lambda_2\,\mathbf{exp}\left(\Lambda_2 z\right)e^{i\Psi} + \text{c.c.} =$$

$$= j_0 + \text{const. } \mathbf{exp}\frac{\sqrt{3}\Gamma z}{2}\cdot\mathbf{exp}\,i\left[\left(\frac{\Gamma}{2}+k_u+k_L\right)z-\omega t\right] + \text{c.c.}$$

Thus, the phase velocity of the density modulation is given by

$$v_{j1} = \frac{\omega}{k_{eff}} = \frac{\omega}{k_u + k_L + \frac{\Gamma}{2}} \approx \frac{\omega}{k_u + k_L} - \frac{\Gamma}{2k_L}c\,.$$

Since $\omega/k_u + k_L$ is the mean longitudinal velocity of the resonant electrons, it is seen that the growing density modulation slowly slips backwards with respect to the bunch centre.

Finally, the group velocity of electromagnetic wave packets during the FEL process is of interest. Analysing how Λ_2 depends on c shows that

$$v_g = \frac{d\omega}{dk} \approx c\left(1 - \frac{1+K^2}{3\gamma_0^2}\right)\,.$$

In conclusion, we can distinguish four characteristic velocity slippages with respect to c in the high-gain FEL:

$c - v_{ph} = \frac{\Gamma}{2k_L}c$ with v_{ph} the phase velocity of on electromagnetic wave during the gain process.

$c - v_g = c\frac{1+K^2}{3\gamma_0^2}$ with v_g the group velocity of on electromagnetic wave during the gain process.

$c - v_z = c\frac{1+K^2}{2\gamma_0^2}$ with v_z the longitudinal velocity of the electron bunch

(i.e. of resonant particles, 'kinematical slippage').

$c - v_{j1} = c\left(\frac{1+K^2}{2\gamma_0^2} + \frac{\Gamma}{2k_L}\right)$ with v_{j_1} the phase velocity of density modulation during the gain process.

From these relations, we can calculate the slippage $v_g - v_z$ of radiation wave packages ('spikes' in the time domain) with respect to the electron bunch. It is three times smaller than the kinematic slippage:

$$\frac{v_g - v_z}{c - v_z} = \frac{1}{3}\,.$$

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REFERENCES

[1] K. Wille, *Introduction to Accelerator Physics* (Teubner Studienbücher, 1992).

[2] E.L. Saldin, E.A. Schneidmiller M.V. Yurkov *The Physics of Free-Electron Lasers* (Springer, 2000).

[3] J.D. Jackson, *Classical Electrodynamics* (De Gruyter, 1980).

[4] P. J. Duke, *Synchrotron Radiation* (Oxford Science Publications, 2000).

[5] J.M.J. Madey, Relationship between mean radiated energy and spontaneous power spectrum in a free electron laser, *Nuovo Cimento* **B50** (1979) 64.

[6] A. Renieri, *Nuovo Cim*. **53B** (1979) 160.

[7] J. Andruszkow *et al.*, First observation of Self-Amlified Spontaneous Emission in a Free-Electron Laser at 109 nm Wavelength, *Phys. Rev. Lett*. **85** (2000) 3825.

[8] A. M. Kondratenko and E. L. Saldin, Generation of coherent radiation by a relativistic electron beam in an undulator, *Part. Accelerators*, **10** (1980) 207.